Battle of Opinions over Evolving Social Networks

Irem Koprulu, Yoora Kim and Ness B. Shroff

***Abstract*—Social networking platforms are major players in the discussion and formation of opinions in diverse areas including, but not limited to, political discourse, market trends, news and social movements. Often, these opinions are of a competing nature, e.g., radical vs. peaceful ideologies, correct information vs. misinformation on a piece of news, one technology vs. another. We study the battle of such competing opinions over evolving social networks. In particular, we model the interactions of multiple opinions over a common networking platform, and characterize how they evolve over a dynamically expanding opinion network. The novelty of our model is that it captures the exposure and adoption dynamics of opinions that account for the preferential and random nature of exposure as well as the persuasion power of different opinions. We provide a complete characterization of the mean opinion dynamics over time as a function of the initial adoption as well as the particular exposure and adoption dynamics. Our analysis, supported by case studies, reveals the key metrics that govern the spread of opinions and establishes the means to engineer the desired impact of an opinion in the presence of other competing opinions. These results can also be used to reverse-engineer an observed "battle of opinions" to estimate the inherent persuasion power of the opinions.**

I. Introduction

Social networks, whether face-to-face or digital, capture the connections and interactions between people on a wide range of platforms. They are a medium for the spread of diverse influences including opinion, information, innovation, riots, biological or computer viruses, and even obesity [1]. As such, social networks play a key role in shaping human behavior.

We are interested in understanding the principles and dynamics of *multiple* influences spreading over an evolving social network. Even though the underlying social network platform might be considered static over a shorter time frame, the "influence" subnetwork in which the new influence originates and spreads is dynamically growing. As an example, we regularly see battles of opinions on social platforms, e.g., Twitter, as a reaction to a piece of possibly controversial news. The information spread and opinion formation start with a set of initial nodes. Over time, followers of these initial nodes are exposed to the news and join the dynamically growing opinion subnetwork of nodes who have heard the news and formed an opinion.

Understanding the fate of such competing opinions over social networks demands new models that capture the spreading and adoption dynamics of different opinions over a common network platform. This motivates us in this work to model and study the spreading dynamics of multiple influences over a growing dynamic network. We require our network model to capture several phenomena, such as a heavy-tailed degree distribution, that are observed in many real-world social networks.

The *degree* or connectivity of a node in a network is the number of its connections. Online social networks such as Twitter have been show to have a heavy-tailed degree distribution [2]. This discovery has ignited renewed interest in the theory of scale-free networks. In a scale-free network, the fraction $P(d)$ of vertices with degree d is proportional to $d^{-\gamma}$, where γ is a constant. Scale-free networks enjoyed a great deal of attention since it was discovered that the degree distribution of the World Wide Web (WWW) obeys a power law [3], [4].

The heavy-tailed degree distribution of scale-free networks necessitates a new random graph model other than the Erdös-Rényi random graph. In an Erdös-Rényi graph, any two nodes are connected with a given probability independently of other connections in the network [5]. Such a *random attachment* model gives rise to a Poisson degree distribution as the number of nodes increases [6]. In [7], the authors propose the *preferential attachment* model as a mechanism that gives rise to a power-law degree distribution. In the preferential attachment model, the probability that a node is connected to a given node is proportional to the degree of the given node. A formal proof for the power-law degree distribution is presented in [8].

Various hybrid models that mix preferential and random attachment have been studied in several scenarios of growing networks, social or otherwise (e.g., [9], [10], [11]). In these works, the authors show that networks evolving according to hybrid random-preferential attachment models exhibit a power-law degree distribution and other desirable properties that mimic social networks (e.g., short average distance, large clustering coefficients and positive degree correlation).

There is a rich history of research on the problem of evolving complex networks (e.g., [12], [13] and references therein) as well as influence propagation on static networks (e.g., [14], [15], [16]). But to the best of our knowledge these topics are always studied individually. In addition, there is very little work that concentrates on influence propagation specifically on scale-free networks. In [17] and [18], the authors study the spread of a single virus in a static network generated according to the preferential attachment model. However, they do not seek to characterize the time evolution of the influence spread; their focus is on conditions that give rise to a persistent epidemic.

In this work, we capture the following phenomena: the preferential versus random nature of attachment of newcomer nodes; the varying power of different types of influence in persuading newcomers to adopt their type; varying responsiveness of newcomers to adopt different influences. In particular, we want to answer the following key questions:

- How do the initial acceptance and the persuasion power of different types of influences affect their evolution and limiting dominance? If the source of influence has limited resources to control the initial acceptance and the persuasiveness, how should it distribute it?
- What is the impact of preferential versus random attachment dynamics on the influence spread? Is the influence spread sensitive or robust to such dynamics?
- Supposing that persuasion parameters are not known a priori, can we infer them based on the observed influence spread? This might be viewed as a means of assessing the quality of an idea/product/etc.

In order to answer these questions, we propose a new mathematical model for influence spread on an evolving network. We perform a discrete-time analysis of the mean system dynamics. Since this analysis yields exact results but limited insight, we also provide a relaxed continuous-time analysis to obtain further insights. Through simulation studies and analytical arguments, we verify the closeness of our continuous-time approximation to the discrete-time exact solution. We then translate our analytical results into insights on the characteristics and essential dynamics of important instances of the problem. In particular, we first look at the spread of two competing opinions in an evolving network. Next, we turn our attention to the adoption of two competing technologies in an evolving network with indifferent populations, focusing on the market size and market shares. These investigations reveal the impact of different attachment and adoption dynamics on the transient and limiting behavior of influence spread.

II. Network Evolution and Influence Propagation Model

In this paper, we study the propagation of multiple competing influences over a dynamically expanding network. To that end, we propose a model where multiple types of influences interact with each other as the network expands with newcomer nodes. This model not only captures the popularity or prominence of the existing nodes as measured by their number of connections or degree (as in preferential attachment models), but also the possible differences in the persuasion power of the influences themselves, which is typically determined by the quality of the product or the strength of the opinion.

A. Network Evolution: Exposure to Opinions

The network evolution starts at time t_0 with $N_0 > 0$ initial nodes and a total degree of $D_0 > 0$. We use $N_{tot}[t]$ and $D_{tot}[t]$ to denote the total number of nodes and total degree at time t, respectively. At the end of each discrete-time period $t \in \{t_0+1, t_0+2, t_0+3, \ldots\}$, a new node arrives[1] and connects to one of the existing nodes in the network. We refer to the node to which the newcomer node connects as the *parent node*.

[1]The model can be readily extended to the case where newcomer nodes arrive at possibly random times $\{T_1, T_2, \ldots\}$ with independent inter-arrival times. In that case, all our results still hold when the network is sampled right after the arrival of a new node.

Our current model accounts for a single parent node for each newcomer node. While this assumption is certainly limiting, it still allows our model to capture many real life scenarios where it is possible to identify a most influential existing node for each newcomer node. One example is singling out the node of *first exposure* as the parent node. In the language of the opinion subnetwork then, we say that node A is connected to node B, if A is first exposed to the opinion via B.

An important factor in determining which one of the existing nodes will be the source of exposure to the newcomer nodes is the visibility of the existing nodes as measured in terms of their connectivity. In the Twitter example, the higher the connectivity of a user, the more likely it is that the next user will hear the news from that particular user. Likewise, for customer reviews a higher number of helpful tags adds to the visibility of a particular review, and Google PageRank determines the visibility of personal blogs and other sites based on the hits they have received so far. In all examples, it is also possible that the next exposure will happen through a randomly selected, rather unassuming node.

In order to capture these dual connection dynamics we adopt a *hybrid* connection model composed of *random* and *preferential attachment*. Each newcomer node choses either the random attachment mode with probability $q \in [0,1]$ or the preferential attachment mode with probability $(1-q)$ independently from the choices of the previous nodes. We refer to the probability q as the *attachment parameter*. In the random attachment mode, the newcomer node attaches to an existing node selected uniformly at random, i.e., each node in the network is chosen with equal probability $1/N_{tot}[t]$. In the preferential attachment mode, each node in the network is chosen with a probability that is proportional to its degree, i.e., if a particular node has degree d then it is chosen with probability $d/D_{tot}[t]$.

B. Influence Propagation: Adoption of Opinion/Color

There are M different influences labeled $1, \ldots, M$ propagating in the network. Influence can refer to a wide range of things including opinions, ideas, innovations or products. In the sequel, we will use the word *color* when referring to these influences. Each node adopts only one out of M colors (hence the name competing influences) at the time it joins the network and does not change its color once adopted.

We assume that a newcomer node connects to the network according to the hybrid attachment model described in Section II-A independently of the colors of the exiting nodes. This presumes that attachments are governed in part by the random behavior of the newcomers and in part by the prominence of the existing nodes, but *not* by the adopted colors of the existing nodes. This assumption is justified in many scenarios where the colors of the existing nodes are not discernable at the time of first connection, but their prominence is readily observable by the newcomer through their number of connections.

Once a newcomer connects to a parent node, it becomes receptive to the influence. The newcomer node is not restricted to adopt the same color as its parent node. The parent node's influence only determines the likelihood of the newcomer node adopting each color, e.g., the node can be more likely to adopt the same color as its parent. In particular, if the parent has color $j \in \{1, \ldots, M\}$, then the newcomer node adopts color $i \in \{1, \ldots, M\}$ with probability p_{ij}, i.e.,

$$p_{ij} = \mathbb{P}(\text{Node adopts color } i | \text{Parent node has color } j),$$

where $0 \leq p_{ij} \leq 1$ for all i and j, and $\sum_i p_{ij} = 1$ for each j. The set of adoption parameters $\{p_{ij}\}$ captures the *persuasion power* of different types of influences. Depending on the type of influence, these parameters may reflect the strength of an opinion or inherent quality of a product.

Although each node adopts a single color, this model can also encompass scenarios where newcomer nodes may adopt *zero* or *multiple* colors. Examples include the newcomer node not subscribing to any of the existing opinions, not buying any product, or buying multiple products. In these cases a new color is assigned to these choices.

We use $N_i[t]$ and $D_i[t]$ to denote the number and the total degree of nodes of color i at time t, where $\sum_{i=1}^{M} N_i[t] = N_{tot}[t]$ and $\sum_{i=1}^{M} D_i[t] = D_{tot}[t]$. In order to facilitate a more compact presentation, we define the state vector

$$\mathbf{X}[t] \triangleq (\mathbf{N}[t], \mathbf{D}[t])^T, \tag{1}$$

in terms of the number of nodes $\mathbf{N}[t] \triangleq (N_1[t], \ldots, N_M[t])$ and degrees $\mathbf{D}[t] \triangleq (D_1[t], \ldots, D_M[t])$. The initial state of the network at time t_0 is given by $\mathbf{X}[t_0] = \mathbf{X}_0$.

III. Mean System Dynamics

In this section, we provide analytical results that describe the mean dynamics of the evolving influence network introduced in Section II. We first derive exact results based on the discrete-time (DT) model. The form of these exact results, however, provides only a limited insight into the effect of the various system parameters on the evolution of the system. In order to achieve further insight, we develop and analyze an approximate continuous-time (CT) model. We use these results to reveal important network formation and influence dynamics in the case studies of the subsequent section.

A. Discrete-Time Mean System Analysis

In this subsection, we provide an exact characterization of the mean behavior of the system dynamics in discrete-time by investigating the conditional mean drift of the system state $\mathbf{X}[t]$ defined in (1). In particular, we obtain a linear system with time-varying coefficients to describe the mean system evolution. These coefficients provide valuable information concerning the impact of the hybrid attachment model and the persuasion power parameters on the spread and the degree distribution of different types of influences. We present our main result regarding the nature of influence spread under such dynamics.

Theorem 1 (Linear Time-Varying DT System Description and Solution)**.** *The one-step time evolution of the mean network state described in Section II is governed by the following time-varying linear difference equation*

$$\mathbb{E}[\mathbf{X}[t+1] - \mathbf{X}[t] \mid \mathbf{X}[t]] = \mathbb{A}[t]\mathbf{X}[t], \tag{2}$$

for $t \in \{t_0, t_0+1, \ldots\}$ and initial condition $\mathbf{X}[t_0] = \mathbf{X}_0$. $\mathbb{A}[t]$ is a $2M \times 2M$ matrix composed of four $M \times M$ constant submatrices $\mathbb{A}_{ij}$, $N_{tot}[t]$ and $D_{tot}[t]$ as follows:

$$\mathbb{A}[t] = \begin{bmatrix} \mathbb{A}_{11}/N_{tot}[t] & 2\mathbb{A}_{12}/D_{tot}[t] \\ \mathbb{A}_{21}/N_{tot}[t] & 2\mathbb{A}_{22}/D_{tot}[t] \end{bmatrix}, \tag{3}$$

where the entries of the constant submatrices are given by

$$\begin{aligned} [\mathbb{A}_{11}]_{i,j} &= qp_{ij}, \\ [\mathbb{A}_{12}]_{i,j} &= \frac{1}{2}(1-q)p_{ij}, \\ [\mathbb{A}_{21}]_{i,j} &= \begin{cases} q(1+p_{ii}), & \text{if } i = j \\ qp_{ij}, & \text{if } i \neq j \end{cases} \\ [\mathbb{A}_{22}]_{i,j} &= \begin{cases} \frac{1}{2}(1-q)(1+p_{ii}), & \text{if } i = j \\ \frac{1}{2}(1-q)p_{ij}, & \text{if } i \neq j. \end{cases} \end{aligned} \tag{4}$$

The mean state of the system at time t is given by

$$\mathbb{E}[\mathbf{X}[t] \mid \mathbf{X}_0] = \left(\prod_{s=t_0}^{t-1} (\mathbb{A}[s] + \mathbb{I}) \right) \mathbf{X}_0, \tag{5}$$

where $\mathbb{I}$ is the $2M \times 2M$ identity matrix.

Proof. The proof is given in Appendix A. □

It is possible, and insightful, to derive a more explicit solution to the general equation governing the network evolution in (5) by imposing a restriction on the initial state of the system. We observe that the total degree in the network $D_{tot}[t] = 2(t - t_0) + D_0$ approaches twice the number of nodes $N_{tot}[t] = (t - t_0) + N_0$ with increasing time t. If we impose the condition $D_0 = 2N_0$ from the onset to ensure $D_{tot}[t] = 2N_{tot}[t]$ for all t, then we can write $\mathbb{A}[t] = \mathbb{A}/(t - t_0 + N_0)$ where $\mathbb{A}$ is the constant matrix composed of the submatrices defined in (4) as follows

$$\mathbb{A} = \begin{bmatrix} \mathbb{A}_{11} & \mathbb{A}_{12} \\ \mathbb{A}_{21} & \mathbb{A}_{22} \end{bmatrix}. \tag{6}$$

The following corollary summarizes our results for this specific case.

Corollary 1. *Provided that $D_0 = 2N_0$, and that the matrix $\mathbb{A}$ is diagonalizable, the expected state of the network described in Section II at time t is given by*

$$\mathbb{E}[\mathbf{X}[t] \mid \mathbf{X}_0] = \mathbb{V}\Lambda[t]\mathbb{V}^{-1}\mathbf{X}_0, \tag{7}$$

where $\Lambda[t]$ is the $2M \times 2M$ diagonal matrix with entries

$$[\Lambda[t]]_{i,i} = \exp\left(\sum_{s=t_0}^{t-1} \log\left(1 + \frac{\lambda_i}{s - t_0 + N_0}\right) \right) \tag{8}$$

and $\{\lambda_i\}_{i=1}^{2M}$ and $\mathbb{V}$ are the eigenvalues and eigenvector matrix of $\mathbb{A}$, respectively.

Proof. The result follows readily from (5) by replacing $\mathbb{A}[s]$ with $\mathbb{A}/(s - t_0 + N_0)$ and $\mathbb{A}$ with $\mathbb{V}\,\mathrm{diag}\left(\{\lambda_i\}_{i=1}^{2M}\right)\mathbb{V}^{-1}$. □

B. Continuous-Time Approximation

In this subsection, we propose a continuous-time approximation to the mean evolution of the influence network. Throughout the paper, we use (t) instead of $[t]$ to distinguish continuous-time variables from their discrete-time counterparts. We introduce the short-hand notation $\mathbf{x}(t) \triangleq \mathbb{E}[\mathbf{X}(t)]$ to denote the CT approximation of the *mean* state vector. Next, we obtain a *heuristic* continuous-time approximation for the evolution of the network by replacing the difference equation in (2) by a differential equation.

Definition 1 (Continuous-Time Approximation of the System State Evolution)**.** *The continuous-time evolution of the mean system state $\mathbf{x}(t)$ is described by the following time-varying linear differential equation:*

$$\frac{d\mathbf{x}(t)}{dt} = \mathbb{A}(t)\mathbf{x}(t), \textit{ for } t \geq t_0, \textit{ and } \mathbf{x}(t_0) = \mathbf{X}_0 \tag{9}$$

where $\mathbb{A}(t)$ has the same form as $\mathbb{A}[t]$ defined in (3)*.*

We derive an explicit solution to the system state evolution in (9) for the case that the initial state satisfies the constraint $D_0 = 2N_0$ as in Corollary 1. In this case, we note that $\mathbb{A}(s)$ commutes with $\mathbb{A}(t)$ for all values of s and t, i.e., $\mathbb{A}(s)\mathbb{A}(t) = \mathbb{A}(t)\mathbb{A}(s)$ for all s, t. The Magnus series [19] consists of a single term and yields the solution given in Corollary 2. Alternatively, we can show that (10) given below solves (9) by direct substitution.

Corollary 2. *When $D_0 = 2N_0$, the solution to* (9) *is given by*

$$\mathbf{x}(t) = \exp\left(\log\left(\frac{t - t_0 + N_0}{N_0}\right)\mathbb{A}\right)\mathbf{X}_0. \tag{10}$$

For diagonalizable $\mathbb{A}$, we can further reduce this solution by substituting the eigendecompostion $\mathbb{A} = \mathbb{V}\,\mathrm{diag}\left(\{\lambda_i\}_{i=1}^{2M}\right)\mathbb{V}^{-1}$ in the definition of the matrix exponential to obtain

$$\mathbf{x}(t) = \mathbb{V}\,\mathrm{diag}\left(\left\{\left(\frac{t - t_0 + N_0}{N_0}\right)^{\lambda_i}\right\}_{i=1}^{2M}\right)\mathbb{V}^{-1}\mathbf{X}_0. \tag{11}$$

Next we argue analytically that the CT approximate solution $\mathbf{x}(t)$ obtained in (11) is indeed a reasonable approximation of the DT exact solution $\mathbf{X}[t]$ obtained in (7). We start by noting that for small x, $\log(1 + x) \approx x$. Hence, for large s, the $\log\left(1 + \frac{\lambda_i}{s - t_0 + N_0}\right)$ terms in (8) can be approximated by $\frac{\lambda_i}{s - t_0 + N_0}$. Further approximating the sum of the harmonic terms by the corresponding integral results in the following approximation for the entries of the diagonal matrix:

$$\begin{aligned}
&\exp\left(\sum_{s=t_0}^{t-1}\log\left(1 + \frac{\lambda_i}{s - t_0 + N_0}\right)\right) \\
&\approx \exp\left(\lambda_i \sum_{s=t_0}^{t-1}\frac{1}{s - t_0 + N_0}\right) \\
&\approx \exp\left(\lambda_i\left(\log(t - t_0 + N_0) - \log(N_0)\right)\right) \\
&= \left(\frac{t - t_0 + N_0}{N_0}\right)^{\lambda_i}
\end{aligned}$$

With this approximation, (7) reduces to (11) as claimed. We also note that the diagonal terms corresponding to zero eigenvalues in the DT solution given in (7) and CT approximate solution given in (11) are an exact match. Hence, only non-zero eigenvalues contribute to the difference between the two solutions.

Finally, we present a continuous-time arrival model under which the differential equation (9) holds exactly. To mimic the linear arrivals in the DT model (i.e., $N_{tot}[t] = (t - t_0) + N_0$ for $t \in \{t_0, t_0 + 1, \ldots\}$), we suppose that the arrival process in the CT model is generated according to

$$\begin{aligned}
&\mathbb{P}(\text{Exact 1 newcomer node arrives during } (k, k+\delta]) = \delta \\
&\mathbb{P}(\text{No newcomer node arrives during } (k, k+\delta]) = 1 - \delta,
\end{aligned}$$

where $k \in \{t_0, t_0 + 1, \ldots\}$ and $\delta \in (0, 1]$. Then, the number of nodes under the CT model at time $t \in \{t_0, t_0 + 1, \ldots\}$ is $N_{tot}(t) = (t - t_0) + N_0$ with probability 1, which is identical to that of the DT model.

For this CT arrival model, we can extend the difference equation (2) in Theorem 1 to

$$\mathbb{E}[\mathbf{X}(t+\delta) - \mathbf{X}(t)|\mathbf{X}(t)] = \delta\mathbb{A}(t)\mathbf{X}(t), \tag{12}$$

for any $t \geq t_0$ and $\delta \in [0, 1]$. Recalling that $\mathbf{x}(t) = \mathbb{E}[\mathbf{X}(t)]$, the expectation of the left-hand side of (12) reduces to

$$\begin{aligned}
\mathbb{E}\big[\mathbb{E}[\mathbf{X}(t+\delta) - \mathbf{X}(t)|\mathbf{X}(t)]\big] &= \mathbb{E}[\mathbf{X}(t+\delta) - \mathbf{X}(t)] \\
&= \mathbb{E}[\mathbf{X}(t+\delta)] - \mathbb{E}[\mathbf{X}(t)] \\
&= \mathbf{x}(t+\delta) - \mathbf{x}(t).
\end{aligned}$$

Hence, the expectation of (12) yields

$$\mathbf{x}(t+\delta) - \mathbf{x}(t) = \delta\mathbb{A}(t)\mathbf{x}(t). \tag{13}$$

Since (13) holds for all $\delta \in (0, 1]$, it follows that

$$\lim_{\delta \to 0}\frac{\mathbf{x}(t+\delta) - \mathbf{x}(t)}{\delta} = \mathbb{A}(t)\mathbf{x}(t),$$

which is the differential equation (9) in our CT approximation.

We have compared both DT and CT results and Monte Carlo simulations of our model for several sets of system parameters. Our results verify that the difference between the DT and CT evolutions is negligible. Fig. 1 depicts the typical discrepancy between the DT and CT solutions for a set of parameters in an evolving network with two opinions.

In the subsequent two sections, we proceed to translate these analytical results into insights on the characteristics and essential dynamics of important instances of the problem. These investigations reveal the impact of different attachment and adoption dynamics on the transient and limiting behavior of influence spread.

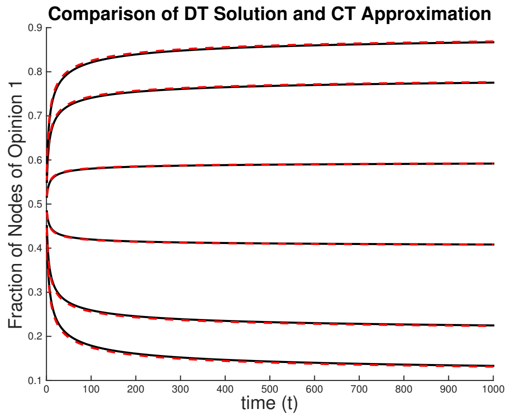

Fig. 1. The DT solution and the CT approximation for the mean fraction of nodes of one opinion in an evolving network with two opinions. The connection parameter has value $q = 0.5$ and the set of adoption parameters are given by $p_{12} \in \{0.05, 0.1, 0.2, 0.3, 0.4, 0.45\}$ and $p_{21} \in \{0.45, 0.4, 0.3, 0.2, 0.2, 0.05\}$. The network evolves from one node of each opinion in all cases.

IV. Battle of Two Opinions

In this section, we present the detailed solution to the continuous-time approximation with two competing influences in the network. Binary systems arise in a vast number of real life scenarios that are based on adopting or rejecting a single opinion, belief, technology or product. The importance of studying the two influence case is not only due to its applicability to these scenarios. Its relative simplicity allows us to gain insights into the dynamics of influence propagation on evolving systems, which can be generalized to scenarios with larger number of influences.

We consider a scenario in which nodes in an evolving network adopt opinion 1 or opinion 2 as described in Section II. The system can be fully described in terms of the initial state $\mathbf{X}_0$, the attachment parameter q, and the two cross-adoption parameters p_{12} and p_{21}. The latter quantify the rate of *defection* from an opinion, i.e. the failure rate of an existing node to persuade newcomer nodes to subscribe to the same opinion as itself. We define $\widetilde{p} = p_{12} + p_{21}$ and exclude the degenerate case of $\widetilde{p} = 0$ from our discussion. In this case, newcomer nodes adopt their parent node's opinion without fail.

We assume, without loss of generality, that the network evolution starts at time $t_0 = 0$. For the initial state $\mathbf{X}_0 = (N_1(0), N_2(0), D_1(0), D_2(0))^T$, we impose the condition that $D_0 = 2N_0$ (i.e., $D_1(0) + D_2(0) = 2(N_1(0) + N_2(0))$) in order to facilitate an algebraic solution. The following is the main result of this case study, which describes the evolution of mean adoption dynamics in terms of initial conditions as well as attachment and influence dynamics.

Theorem 2. *For the network evolution and influence propagation dynamics described above, the continuous-time approximation to the mean number of nodes $n_i(t) = \mathbb{E}[N_i(t)]$ adopting each opinion is given by*

$$\begin{aligned} n_1(t) &= \alpha_1(t + N_0) + \beta\left(\frac{t + N_0}{N_0}\right)^{\lambda} + \gamma, \\ n_2(t) &= \alpha_2(t + N_0) - \beta\left(\frac{t + N_0}{N_0}\right)^{\lambda} - \gamma, \end{aligned} \tag{14}$$

where the coefficients α_i, β, γ and the exponent λ depend on the system parameters as follows:

$$\begin{aligned} &\lambda = 1 - \frac{1}{2}(1+q)\widetilde{p}, \quad \alpha_1 = \frac{p_{12}}{\widetilde{p}}, \quad \alpha_2 = \frac{p_{21}}{\widetilde{p}}, \\ &\beta = \frac{2(1-\widetilde{p})(p_{21}N_1(0) - p_{12}N_2(0))}{\widetilde{p}(2-(1+q)\widetilde{p})}, \\ &\gamma = \frac{(1-q)(p_{21}N_1(0) - p_{12}N_2(0))}{2-(1+q)\widetilde{p}}. \end{aligned}$$

Proof. The proof is given in Appendix B. □

Several observations can be made concerning the evolution of the mean number of nodes adopting each opinion.

Linear and Sublinear Terms in the Evolution: The first term in each expression indicates a *linear* growth of the mean number of nodes with time. The exponent that governs the second terms is common, and satisfies $\lambda \in [-1, 1]$ for all system dynamics. The extreme case of $\lambda = 1$ is achieved only when $\widetilde{p} = 0$. Hence, the second term is *sublinear* and will eventually be dominated by the linear first term. It is also interesting to observe that λ can take negative values, in which case the contribution of the second terms vanish with t.

Long-Term Adoption Characteristics: In view of the previous observation, as long as the defection rate $\widetilde{p} > 0$, the long-term adoption of an opinion is dominated by the linearly increasing component of the evolution. In particular, the fractions of the two opinions in the network converge to $\alpha_1 = p_{12}/\widetilde{p}$ and $\alpha_2 = p_{21}/\widetilde{p}$, respectively. *Thus, the long-term market share of a product is not influenced by the attachment dynamics (as captured by q) or the initial number of the early adopters (as captured by $\mathbf{X}_0$)), but solely by the persuasiveness of the opinions (as captured by cross-adoption probabilities p_{12}, p_{21}).* Fig. 2 confirms this long-term behavior by showing that the fraction of two opinions converges to the same limit for different values of q.

Impact of Attachment Model on the Evolution: Despite the dominance of the linear term in the long-term, the sublinear terms associated with the exponent λ and the coefficient β may have non-negligible *short-term* effects. In fact, such short-term characteristics may be of greater interest for many scenarios in which the influence spread occurs over a short/moderate lifetime. Here, we first observe that the exponent λ increases both with decreasing defection rate $\widetilde{p}$ and with decreasing randomness of attachment q. In other words, as the attachment model tends more towards pure preferential attachment, i.e., q decreases towards 0, the short-term effects are more pronounced in the exponent. Fig. 2 depicts this effect. The evolution curves with $q = 0$ corresponding to pure preferential attachment approach the limiting ratios α_1 and α_2 more slowly.

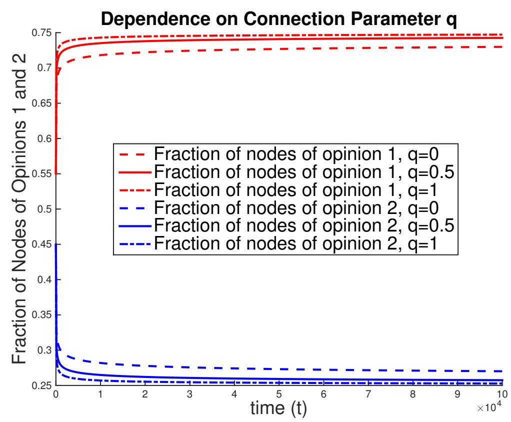

Fig. 2. The impact of the attachment parameter q on the mean fraction of nodes in an evolving network with two opinions. The attachment parameter varies as $q \in \{0, 0.5, 1\}$, while the cross-adoption parameters are fixed as $p_{12} = 0.3$ and $p_{21} = 0.1$ resulting in the limits $\alpha_1 = 0.75$ and $\alpha_2 = 0.25$. The upper set of curves depicts the fraction of nodes adopting opinion 1, while the lower set of curves depicts the fraction of nodes adopting opinion 2.

Impact of Initial Adopters on the Evolution: The coefficient β of the sublinear term depends on the composition of the early adopters as well as the cross-adoption probabilities. The effect of the initial network composition on the evolution of the system is through this coefficient only. Fig. 3 depicts the results of Monte Carlo simulations. First, we note how in accordance with the previous observations the long-term limits of α_1 and α_2 are unaffected by the initial network composition. We also observe that even starting from an extreme initial condition, i.e., all initial nodes of a single opinion, the expected fraction of nodes of each opinion reaches an equilibrium in relatively short time.

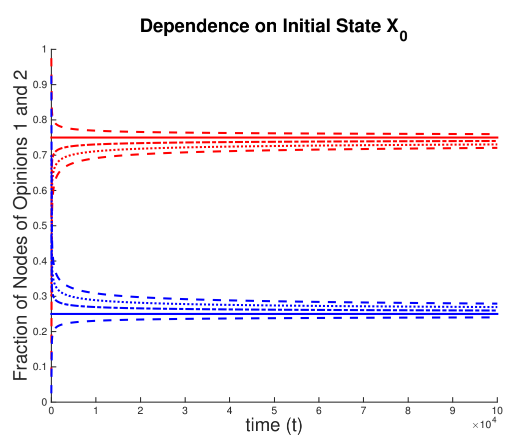

Fig. 3. The dependence of the mean fraction of nodes in an evolving network with two opinions on the initial state $\mathbf{X}_0$of the network. The upper set of curves depicts the fraction of nodes adopting opinion 1, while the lower set of curves depicts the fraction of nodes adopting opinion 2. Individual curves show the evolution starting with varying ratios $\{0, 0.25, 0.5, 0.75, 1\}$ of nodes of each opinion. The attachment parameter is $q = 0.5$ and the adoption parameters are $p_{12} = 0.3$, $p_{21} = 0.1$ resulting in the limits $\alpha_1 = 0.75$ and $\alpha_2 = 0.25$.

The above observations suggest an interesting *connection between attachment dynamics and the early spread of an influence*. In particular, the emergence of prominent (well-connected) members in a society as determined by the attachment dynamics allows the initial influence of the early adopters to survive longer. More specifically, as the q parameter decreases, the degree distribution has heavier tails, thereby indicating emergence of influential/prominent agents. The color of these prominent members will be shaped by the initial composition of the network, which, in turn, will sustain these early impacts for increasingly longer time frames depending on the value of λ. Yet, our model also reveals that the *long-term spread of two competing opinions is ultimately governed by their inherent strengths*.

V. Two Competing Technologies in a Network with Indifferent Populations

In this section, we study the dynamics of innovation spread in the case of two competing alternatives in an evolving network where nodes are allowed to remain *indifferent*, i.e., they adopt neither of the two technologies. We model a word-of-mouth marketing scenario in which newcomer nodes are exposed to the innovation only if their parent node has adopted one of the technologies. In that case, they can adopt one of the innovations (including the competitor of the technology adopted by the parent node) or they can remain indifferent. Indifferent nodes, on the other hand, play a special role in this model in which they do not expose newcomer nodes to either innovation. Any newcomer node that connects to an indifferent node remains indifferent with probability 1.

With the presence of indifferent nodes, not every node partakes in adopting the innovation. This is in contrast to the model in Section IV where each node actively participated in the battle of opinions. With indifferent nodes in the network, we are interested both in the individual number of adopters of each technology and the size of the entire market.

In the language of Section II, we have *three* colors: adopting one of the two technologies (labeled colors 1 and 2) and remaining indifferent (labeled color 3). Given the adoption dynamics described above the system can be fully described by the attachment parameter q and the adoption parameters p_{11}, p_{12}, p_{21} and p_{22}. (Note that $p_{13} = p_{23} = 0, p_{33} = 1$, $p_{31} = 1 - (p_{11} + p_{21})$ and $p_{32} = 1 - (p_{12} + p_{22})$.) As in the previous section, we assume, without loss of generality, that the network evolution starts at time $t_0 = 0$. We also assume that the initial number of nodes and initial total degree in the network satisfy $D_0 = 2N_0$.

Theorem 3. *For the network evolution and influence propagation dynamics described above, the continuous-time approximation to the mean number of nodes $n_i(t) = \mathbb{E}[N_i(t)]$ adopting each technology is given by*

$$\begin{aligned} n_1(t) &= \alpha_1 \left(\frac{t+N_0}{N_0}\right)^{\lambda_1} + \beta_1 \left(\frac{t+N_0}{N_0}\right)^{\lambda_2} + \gamma_1, \\ n_2(t) &= \alpha_2 \left(\frac{t+N_0}{N_0}\right)^{\lambda_1} + \beta_2 \left(\frac{t+N_0}{N_0}\right)^{\lambda_2} + \gamma_2, \end{aligned} \tag{15}$$

while the mean number of indifferent nodes is

$$n_3(t) = t + N_0 - n_1(t) - n_2(t).$$

The coefficients $\alpha_i \geq 0, \beta_i, \gamma_i$ are constants that depend on the system parameters $p_{11}, p_{12}, p_{21}, p_{22}, q$ and initial state $\mathbf{X}_0$. The exponents λ_1 and λ_2 are given by

$$\begin{aligned} \lambda_1 &= \frac{1}{2}(1-q) + \frac{1}{4}(1+q)(p_{11} + p_{22} + \Delta), \\ \lambda_2 &= \frac{1}{2}(1-q) + \frac{1}{4}(1+q)(p_{11} + p_{22} - \Delta), \end{aligned} \tag{16}$$

where $\Delta = \sqrt{(p_{11} - p_{22})^2 + 4p_{12}p_{21}}$. The exponents satisfy $\lambda_2 \leq \lambda_1 \leq 1$ and the latter equality holds if and only if

$$p_{11} + p_{21} = p_{12} + p_{22} = 1. \tag{17}$$

Proof. The derivation of (15) and (16) is similar to the proof of Theorem 2 given in Appendix B. Hence, we omit the details. To establish the range of the exponents, we note that

$$\begin{aligned} \Delta &= \sqrt{(p_{11} - p_{22})^2 + 4p_{12}p_{21}} \\ &\leq \sqrt{(p_{11} - p_{22})^2 + 4(1 - p_{22})(1 - p_{11})} \quad (18) \\ &= \sqrt{((p_{11} + p_{22}) - 2)^2} = 2 - p_{11} - p_{22}. \quad (19) \end{aligned}$$

Hence, we obtain the bound $p_{11} + p_{22} + \Delta \leq 2$ and conclude that $\lambda_1 \leq 1$. Note that (18) is met with equality if and only if (17) is satisfied. Therefore, $\lambda_1 = 1$ if and only if (17) holds. □

The dependence of the coefficients $\alpha_i, \beta_i, \gamma_i$ in Theorem 3 on the system parameters $\{p_{ij}\}, q$ and $\mathbf{X}_0$ is quite complex. For the case of $q = 1$ corresponding to pure random attachment, we have

$$\begin{aligned} \alpha_1 &= \frac{1}{2\Delta}\left((p_{11} - p_{22} + \Delta)N_1(0) + 2p_{12}N_2(0)\right), \\ \alpha_2 &= \frac{1}{2\Delta}\left(2p_{21}N_1(0) + (-p_{11} + p_{22} + \Delta)N_2(0)\right), \\ \beta_1 &= \frac{1}{2\Delta}\left((-p_{11} + p_{22} + \Delta)N_1(0) - 2p_{12}N_2(0)\right), \\ \beta_2 &= \frac{1}{2\Delta}\left((-2p_{21}N_1(0) + (p_{11} - p_{22} + \Delta)N_2(0)\right), \\ \gamma_1 &= \gamma_2 = 0. \end{aligned}$$

Several observations can be made concerning the evolution of the mean number of nodes adopting each technology and can be contrasted to the two color case without indifferent nodes.

Sublinear Growth of the Market Size: We note that only the expression for the mean number of indifferent nodes $n_3(t)$ has a linear term. The mean number of nodes adopting one of the two active influences is governed by the sublinear t^{λ_1} term. According to Theorem 3, $\lambda_1 < 1$ unless nodes exposed to either form of innovation do not have the option of remaining indifferent. We omit this case from the discussion below. As a result, the fraction of each active influence within the total network population tends to zero in the long term. Nevertheless, there are two important measures to be studied: the total number of nodes adopting a new technology and the fraction of each technology among these nodes, i.e., the market size and the market share.

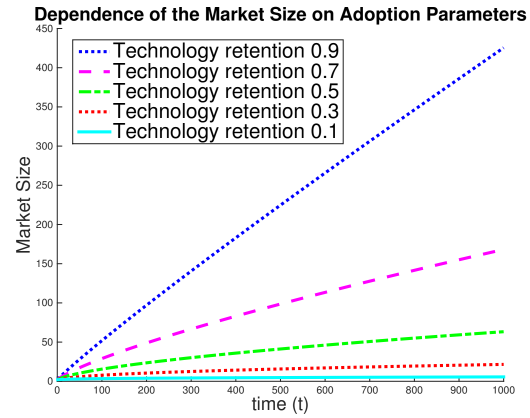

Fig. 4. The impact of the adoption parameters $\{p_{ij}\}$ on the market size in an evolving network with indifferent nodes. The technology retention probability $p_{11} + p_{21} = p_{12} + p_{22}$ varies as $\{0.1, 0.3, 0.5, 0.7, 0.9\}$. The fraction of nodes that opt for the same technology are 80% and 70%, respectively, for technologies 1 and 2. The attachment parameter is fixed as $q = 0.5$ and the network evolution starts with one node of each color.

Impact of Adoption Model on the Market Size: The size of the market is given by the total number of nodes adopting a new technology, i.e., $n_1(t) + n_2(t)$. While the market size is affected by all system parameters, the largest effect is due to the adoption parameters $\{p_{ij}\}$. In particular, the market grows monotonically with growing sums $p_{11} + p_{21}$ and $p_{12} + p_{22}$, as these sums represent the probability that a node exposed to the innovation does adopt either form of it. We call this measure the *technology retention* probability. Fig. 4 depicts the growth of the market with increasing technology retention probability.

Impact of Attachment Model on the Market Size: The growth of the market size is dominated by the $(\alpha_1 + \alpha_2)t^{\lambda_1}$ term. Hence, the largest impact of the attachment model on the market size, especially in the long term, is through the dependence of the exponent λ_1 on the attachment parameter q. In light of (19), $\lambda_1 = \frac{1}{4}(p_{11} + p_{22} + \Delta + 2) + \frac{1}{4}q(p_{11} + p_{22} + \Delta - 2)$ is a linearly decreasing function of q for all sets of adoption parameters $\{p_{ij}\}$. As a result, the market size grows as the rate of random attachment q decreases. A higher rate of preferential attachment allows individual nodes to establish higher prominence. Early technology adopters develop high degree, which attracts more of the newcomer nodes to one of the technologies, resulting in a larger market. Fig. 5 visualizes this effect of the attachment parameter q on the market size.

Long-Term Market Share Characteristics: The long-term market share of each technology is determined by the coefficients α_1 and α_2 of the t^{λ_1} term in (15). These coefficients depend not only on the adoption parameters $\{p_{ij}\}$ but also on the attachment parameter q and the initial state of the network $\mathbf{X}_0$. This dependence is in apparent contrast to the previous case of two opinions presented in Section IV, where the leading coefficients α_1 and α_2 in (14) depended only on the adoption parameters. Nevertheless, the long-term market share of each product is not influenced by the attachment dynamics

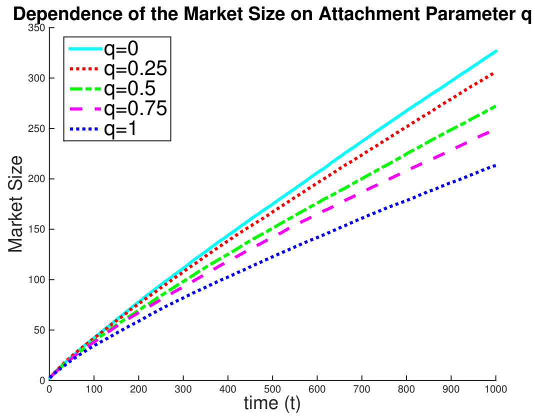

Fig. 5. The impact of the attachment parameter q on the market size in an evolving network with indifferent nodes. The attachment parameter varies as $q \in \{0, 0.25, 0.5, 0.75, 1\}$, while the adoption parameters are fixed as $p_{11} = 0.6, p_{21} = 0.3, p_{12} = 0.1$ and $p_{22} = 0.8$ The network evolution starts with one node of each color.

(as captured by q) nor the initial number of the early adopters (as captured by $\mathbf{X}_0$). In particular, the long-term fraction of technology 1 within the market is given as follows (product 2 occupies the remaining fraction of the market):

$$\frac{\alpha_1}{\alpha_1 + \alpha_2} = \frac{p_{11} - 2p_{12} + \Delta - p_{22}}{2(p_{11} - p_{12} + p_{21} - p_{22})}.$$

Consequently, the attachment model and the preferences of the initial adopters have only short-term effects on the market share. In the long term, the effect of the adoption parameters dominates. Fig. 6 demonstrates how the effect of the initial network composition on the evolution of the market shares diminishes with time.

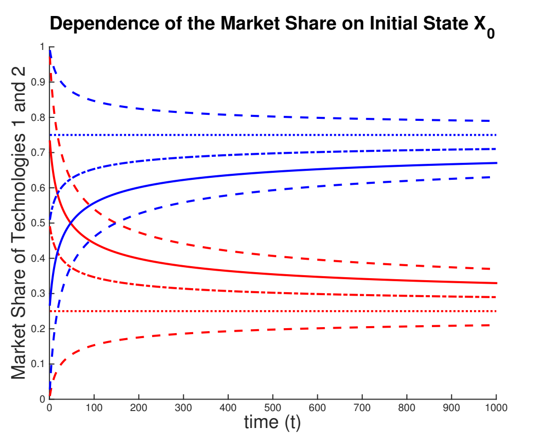

Fig. 6. The dependence of the market share on the initial network state. The ratio of the initial adopters of the two technologies vary as $\{0, 0.25, 0.5, 0.75, 1\}$. The attachment parameter is fixed as $q = 1$, the adoption parameters are fixed as $p_{11} = 0.6, p_{21} = 0.3, p_{12} = 0.1$ and $p_{22} = 0.8$. The network evolution starts with one node of each color.

These observations reiterate the suggestion that the *long-term spread of two competing influences is ultimately governed by their inherent strength*. The attachment dynamics and the early adopters have only a secondary effect on the market size and market share.

VI. Conclusion

In this paper, we have introduced a new analytical model to study the battle of opinions over social networking platforms. In particular, we focused on the spread of multiple competing influences over a simultaneously evolving network. This simple yet powerful model, has allowed us to capture a range of exposure and adoption dynamics, which account for both the preferential and random nature of exposure, as well as different persuasion power of different opinions. We have analytically characterized the evolution of the mean influence spread over time as a function of the initial adoption, as well as, exposure and adoption dynamics. Our analysis, supported by two case studies for further insights, has shown that the persuasion power of an influence has the most potent effect on the extend of its spread. We have further observed how exposure dynamics determine whether the initial adopters plays a short or long lived effect on the evolution of the influence spread. Our work has provided a useful new model with several potential directions for extension.

Appendix A
Proof of Theorem 1

The number of nodes of color i increases by one when the newcomer node adopts color i, regardless of the color of the parent node. Thus, the mean change in the number of nodes of color i from time t to $t+1$ is captured by the following difference equation:

$$\begin{aligned}
&\mathbb{E}[N_i[t+1] - N_i[t] \mid \mathbf{X}[t]] \\
&= \sum_{j=1}^{M} \mathbb{P}\left(\begin{array}{c} \text{New node connects to color } j \\ \text{and adopts color } i \end{array} \mid \mathbf{X}[t] \right) \\
&= \sum_{j=1}^{M} p_{ij} \mathbb{P}\left(\text{New node connects to color } j \mid \mathbf{X}[t]\right) \\
&= \sum_{j=1}^{M} p_{ij} \left(q \frac{N_j[t]}{N_{tot}[t]} + (1-q) \frac{D_j[t]}{D_{tot}[t]} \right). \qquad (20)
\end{aligned}$$

Next, to quantify the mean change in the total degree of the nodes of color i, note that the total degree of the nodes of color i increases by 2 when the newcomer node connects to an existing node of color i and adopts color i, while it increases only by 1 when the newcomer node connects to a node of color i but adopts another color, or when the newcomer node connects to a node of another color and adopts color i. This translates into the following conditional mean drift expression:

$$\begin{aligned}
&\mathbb{E}[D_i[t+1] - D_i[t] \mid \mathbf{X}[t]] \\
&= 2\mathbb{P}\left(\begin{array}{c} \text{New node connects to color } i \\ \text{and adopts color } i \end{array} \mid \mathbf{X}[t] \right) \\
&\quad + \sum_{j \neq i} \mathbb{P}\left(\begin{array}{c} \text{New node connects to color } i \\ \text{and adopts color } j \end{array} \mid \mathbf{X}[t] \right) \\
&\quad + \sum_{j \neq i} \mathbb{P}\left(\begin{array}{c} \text{New node connects to color } j \\ \text{and adopts color } i \end{array} \mid \mathbf{X}[t] \right)
\end{aligned}$$

$$
\begin{aligned}
&= 2p_{ii}\left(q\frac{N_i[t]}{N_{tot}[t]} + (1-q)\frac{D_i[t]}{D_{tot}[t]}\right) \\
&\quad + \sum_{j\neq i} p_{ji}\left(q\frac{N_i[t]}{N_{tot}[t]} + (1-q)\frac{D_i[t]}{D_{tot}[t]}\right) \\
&\quad + \sum_{j\neq i} p_{ij}\left(q\frac{N_j[t]}{N_{tot}[t]} + (1-q)\frac{D_j[t]}{D_{tot}[t]}\right) \\
&= (1+p_{ii})\left(q\frac{N_i[t]}{N_{tot}[t]} + (1-q)\frac{D_i[t]}{D_{tot}[t]}\right) \\
&\quad + \sum_{j\neq i} p_{ij}\left(q\frac{N_j[t]}{N_{tot}[t]} + (1-q)\frac{D_j[t]}{D_{tot}[t]}\right), \qquad (21)
\end{aligned}
$$

where the last equality follows from the fact that $\sum_{i=1}^{M} p_{ij} = 1$. Combining (20) and (21) yields (2).

To obtain the solution to equation (2), we proceed as follows:

$$
\begin{aligned}
&\mathbb{E}[\mathbf{X}[t] \mid \mathbf{X}_0] - \mathbf{X}_0 \\
&= \sum_{s=0}^{t-1} \mathbb{E}[\mathbf{X}[s+1] - \mathbf{X}[s] \mid \mathbf{X}_0] \qquad (22) \\
&= \sum_{s=0}^{t-1} \mathbb{E}[\mathbb{E}[\mathbf{X}[s+1] - \mathbf{X}[s] \mid \mathbf{X}[s], \mathbf{X}_0] \mid \mathbf{X}_0] \qquad (23) \\
&= \sum_{s=0}^{t-1} \mathbb{A}[s]\mathbb{E}[\mathbf{X}[s] \mid \mathbf{X}_0]. \qquad (24)
\end{aligned}
$$

We express the mean state at time t as a telescoping sum of one-step mean drifts in equation (22). Equation (23) follows from the law of total expectation. We obtain equation (24) by applying the drift expression obtained in (2) to the inner expectation.

Finally, we arrive at the iterative expression for the mean state by grouping terms together. The final result (5) is obtained by iterating the last equation below:

$$
\begin{aligned}
\mathbb{E}[\mathbf{X}[t] \mid \mathbf{X}_0] &= \mathbf{X}_0 + \sum_{s=0}^{t-1} \mathbb{A}[s]\mathbb{E}[\mathbf{X}[s] \mid \mathbf{X}_0] \\
&= \mathbf{X}_0 + \sum_{s=0}^{t-2} \mathbb{A}[s]\mathbb{E}[\mathbf{X}[s] \mid \mathbf{X}_0] \\
&\qquad + \mathbb{A}[t-1]\mathbb{E}[\mathbf{X}[t-1] \mid \mathbf{X}_0] \\
&= (\mathbb{I} + \mathbb{A}[t-1])\mathbb{E}[\mathbf{X}[t-1] \mid \mathbf{X}_0].
\end{aligned}
$$

APPENDIX B
PROOF OF THEOREM 2

Throughout this proof we use the complement notation $\overline{c} \triangleq 1 - c$ in order to facilitate a compact notation.

The continuous-time evolution of the mean state is given by

$$
\frac{d}{dt}\mathbf{x}(t) = \frac{1}{t + N_0}\mathbb{A}\mathbf{x}(t), \text{ for } t \geq 0,
$$

and $\mathbf{x}(0) = \mathbf{X}_0$, where $\mathbb{A}$ is the constant matrix given by

$$
\mathbb{A} = \begin{bmatrix}
q\overline{p}_{21} & qp_{12} & \frac{1}{2}\overline{q}\cdot\overline{p}_{21} & \frac{1}{2}\overline{q}p_{12} \\
qp_{21} & q\overline{p}_{12} & \frac{1}{2}\overline{q}p_{21} & \frac{1}{2}\overline{q}\cdot\overline{p}_{12} \\
q(1+\overline{p}_{21}) & qp_{12} & \frac{1}{2}\overline{q}(1+\overline{p}_{21}) & \frac{1}{2}\overline{q}p_{12} \\
qp_{21} & q(1+\overline{p}_{12}) & \frac{1}{2}\overline{q}p_{21} & \frac{1}{2}\overline{q}(1+\overline{p}_{12})
\end{bmatrix}.
$$

The matrix $\mathbb{A}$ is diagonalizable and has eigenvalues $\{1, 0, 0, \lambda = 1 - \frac{1}{2}(1+q)\widetilde{p}\}$. After an eigendecomposition, we evaluate the expression in (9) to obtain the results.

REFERENCES

[1] N. A. Christakis and J. H. Fowler, "The spread of obesity in a large social network over 32 years," *New England Journal of Medicine*, vol. 357, no. 4, pp. 370–379, 2007.

[2] H. Kwak, C. Lee, H. Park, and S. Moon, "What is twitter, a social network or a news media?" in *Proceedings of the 19th International Conference on World Wide Web*, ser. WWW '10. New York, NY, USA: ACM, 2010, pp. 591–600.

[3] R. Albert, H. Jeong, and A. Barabási, "Diameter of the world-wide web," *Nature*, vol. 401, no. 6749, pp. 130–131, 1999.

[4] R. Kumar, P. Raghavan, S. Rajagopalan, D. Sivakumar, A. Tomkins, and E. Upfal, "Stochastic models for the web graph," in *Proceedings of the 41st Annual Symposium on Foundations of Computer Science*, ser. FOCS '00. Washington, DC, USA: IEEE Computer Society, 2000, pp. 57–.

[5] P. Erdös and A. Rényi, "On random graphs, I," *Publicationes Mathematicae*, vol. 6, pp. 290–297, 1959.

[6] M. E. J. Newman, S. H. Strogatz, and D. J. Watts, "Random graphs with arbitrary degree distributions and their applications," *Phys. Rev. E*, vol. 64, p. 026118, Jul 2001.

[7] A.-L. Barabasi and R. Albert, "Emergence of scaling in random networks," *Science*, vol. 286, no. 5439, pp. 509–512, 1999.

[8] B. Bollobás, O. Riordan, J. Spencer, and G. Tusnády, "The degree sequence of a scale-free random graph process," *Random Structures and Algorithms*, vol. 18, no. 3, pp. 279–290, 2001.

[9] D. M. Pennock, G. W. Flake, S. Lawrence, E. J. Glover, and C. L. Giles, "Winners don't take all: Characterizing the competition for links on the web," *Proceedings of the National Academy of Sciences*, vol. 99, no. 8, pp. 5207–5211, 2002.

[10] A. Vázquez, "Growing network with local rules: Preferential attachment, clustering hierarchy, and degree correlations," *Phys. Rev. E*, vol. 67, p. 056104, May 2003.

[11] M. O. Jackson and B. W. Rogers, "Meeting strangers and friends of friends: How random are social networks?" *American Economic Review*, vol. 97, no. 3, pp. 890–915, 2007.

[12] S. Shakkottai, M. Fomenkov, R. Koga, D. Krioukov, and K. Claffy, "Evolution of the internet as-level ecosystem," in *Complex Sciences*, ser. Lecture Notes of the Institute for Computer Sciences, Social Informatics and Telecommunications Engineering, J. Zhou, Ed. Springer Berlin Heidelberg, 2009, vol. 5, pp. 1605–1616.

[13] R. Albert and A.-L. Barabási, "Statistical mechanics of complex networks," *Rev. Mod. Phys.*, vol. 74, no. 1, pp. 47–97, Jan. 2002.

[14] J. Kleinberg, R. Kumar, P. Raghavan, S. Rajagopalan, and A. Tomkins, "The web as a graph: Measurements, models, and methods," in *Computing and Combinatorics*, ser. Lecture Notes in Computer Science, T. Asano, H. Imai, D. Lee, S.-i. Nakano, and T. Tokuyama, Eds. Springer Berlin Heidelberg, 1999, vol. 1627, pp. 1–17.

[15] D. Kempe, J. Kleinberg, and E. Tardos, "Maximizing the spread of influence through a social network," in *Proceedings of the Ninth ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*, ser. KDD '03. New York, NY, USA: ACM, 2003, pp. 137–146.

[16] D. Acemoglu, A. Ozdaglar, and E. Yildiz, "Diffusion of innovations in social networks," in *Decision and Control and European Control Conference (CDC-ECC), 2011 50th IEEE Conference on*, Dec 2011, pp. 2329–2334.

[17] N. Berger, C. Borgs, J. T. Chayes, and A. Saberi, "On the spread of viruses on the internet," in *Proceedings of the Sixteenth Annual ACM-SIAM Symposium on Discrete Algorithms*, ser. SODA '05. Philadelphia, PA, USA: Society for Industrial and Applied Mathematics, 2005, pp. 301–310.

[18] R. Pastor-Satorras and A. Vespignani, "Epidemic spreading in scale-free networks," *Phys. Rev. Lett.*, vol. 86, pp. 3200–3203, Apr 2001.

[19] W. Magnus, "On the exponential solution of differential equations for a linear operator," *Communications on Pure and Applied Mathematics*, vol. 7, no. 4, pp. 649–673, 1954.