Optimizing Sampling for Data Freshness: Unreliable Transmissions with Random Two-way Delay

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***Abstract*—In this paper, we study a sampling problem in which fresh samples of a signal (source) are sent through an unreliable channel to a remote estimator, and acknowledgments are sent back over a feedback channel. Both the forward and feedback channels are subject to random transmission times. Motivated by distributed sensing, the estimator can estimate the real-time value of the source signal by combining the signal samples received through the channel and noisy signal observations collected from a local sensor. We prove that the estimation error is a non-decreasing function of the Age of Information (AoI) for received signal samples and design an optimal sampling strategy that minimizes the long-term average estimation error. The optimal sampler design follows a threshold strategy: If the last transmission was successful, the source waits until the expected estimation error upon delivery exceeds a threshold and then sends out a new sample. If the last transmission fails, the source immediately sends out a new sample without waiting. The threshold is the unique root of a fixed-point equation and can be solved with low complexity (e.g., by bisection search). In addition, the proposed sampling strategy is also optimal for minimizing the long-term average of general non-decreasing functions of the AoI. Its optimality holds for general transmission time distributions of the forward and feedback channels.**

I. Introduction

Timely updates are crucial in a large number of applications such as vehicular networks, wireless sensor networks, and UAV navigations. To achieve timely updates, we require the destination of the network to receive the fresh information from the remote source as quickly as possible. The information freshness is measured by age of information, or simply age, which has been widely explored in recent years (e.g., [1]–[21]). Age of information with the function of current time t is defined as $t - U_t$, where U_t is the generation time of the freshest information data. In several different queuing systems, the Last-Generated, First-Served (LGFS) policy is shown to achieve age-optimality [18]–[20]. Scheduling policies in various wireless networks are studied to minimize age [11]–[13], [16], [17]. A literature review of the recent studies in age of information is provided in [3].

Recently, a connection between age of information and remote estimation of time-varying processes (e.g., Wiener process or Ornstein-Uhlenbeck (OU) process) was established [6], [10], [15]. One of the remote estimation objectives in these early studies is to design an optimal sampling policy to minimize the long-term average minimum mean square error (MMSE). The MMSE is equal to a function of age if the sampling policy is independent of the signal being sampled. However, among these studies, the estimator obtains the signal samples that are subject to delay, but neglects the instant noisy observation. For example, in vehicular networks, the estimator can estimate a signal via both the exact signal samples from the remote sensor and the instant camera streaming from the close vehicle sensor over time. To consider both the delayed signal samples and the instant noisy observations, we will apply the Kalman Filter [22] and study the relationship between the new MMSE and age of information.

The desire for timely updates and the study of the new remote estimation problem motivate us to design an optimal sampling policy for minimizing general nonlinear age functions. To reduce the age, the source may need to wait before submitting a new sample [2]. The study in [4] generalizes the result in [2], proposes an optimal sampling policy under a Markov channel with sampling rate constraint, and observes that the zero-wait policy is far from optimal if, for example, the transmission times are heavy-tail distributed or positively correlated. In [5], the authors provide a survey of the age penalty functions that are related to autocorrelation, remote estimation, and mutual information. The optimal sampling solution is a deterministic or randomized threshold policy based on the objective value and the sampling rate constraint. However, in real-time network systems, not only the forward direction but also the feedback direction have a random delay. Such a random two-way delay model is considered in e.g., [8], [9]. In [9], the paper provides a low complexity algorithm with a quadratic convergence rate to compute the optimal threshold. In [8], an optimal joint cost-and-AoI minimization solution is provided for multiple coexisting source-destination pairs with heterogeneous AoI penalty functions. Although the above studies have optimized sampling strategies, they assume that the transmission process is reliable. However, due to the channel fading, the channel conditions are time-varying and thus the transmission process is unreliable.

While most past works in optimizing sampling consider reliable transmissions, the studies in [7], [21] are the exceptions that consider unreliable transmissions. In [7], the authors consider quantization errors, noisy channel, and non-zero receiver processing time, and they establish the relationship between the MMSE and age. For general age functions, they provide the optimal sampling policies given that the sampler needs to wait before receiving feedback. When the sampler does not need to wait, they provide the enhanced sampling policies that perform better than the previous ones. In [21], the authors choose *idle* or *transmit* at each time slot to minimize joint age penalty and transmission cost. The optimality of a threshold-based policy is shown, and the threshold of the policy is computed efficiently. Nevertheless, transmission delays are random rather than constant because of the congestions, random sample size, etc.

Thus, in this paper, we focus our investigation on studying optimal sampling in wireless networks under the following more realistic (and general) conditions that have largely been unexplored: unreliable transmissions and random delay in both forward and feedback directions. Early studies on optimizing sampling assuming reliable channels with random delays have shown that the sampling problem is decomposed into a per-sample problem. The per-sample problem can be further solved by convex optimization (e.g., [2], [4], [5], [9]) or optimal stopping rules (e.g., [6], [10], [15]). Similarly, our problem assuming an unreliable channel is equivalent to a per-epoch problem containing multiple samples until the successful packet delivery. Therefore, the per-epoch problem is still a Markov Decision Process (MDP) with an uncountable state space, which is the key difference with past works, e.g., [2], [4]–[6], [9], [10], [15] and faces the curse of dimensionality. We further compare our technical differences with past works in Section V-C. The main contributions of this paper are stated as follows:

- We first formulate the problem where the estimator estimates a signal in real-time by combining noisy signal observations from a local sensor and accurate signal samples received from a remote sensor. We show that if the sampling policy is made independently of the signal being sampled, the MMSE equals an increasing function of the age of received signal samples.
- For general nonlinear age functions, or simply age penalty functions, we then provide an exact solution for minimizing these data freshness metrics. The optimal sampling policy has a simple threshold-type structure, and the threshold, which can be efficiently computed by bisection search and fixed-point iterations, is equal to the optimal objective value to our problem. An interesting property has been stated: if the previous transmission is successful, the optimal policy may wait for a positive time period before generating the next sample and sending it out; otherwise, no waiting time should be added. The technical proofs to our results are as follows: (i) The value function of the policy proposed is a solution to the Bellman equation. (ii) Under the contraction mapping assumption, the solution to the Bellman equation is unique, which guarantees the optimality of our proposed policy. Our results apply to general monotone age penalty functions, arbitrary probability of transmission failure, and general delay distributions of both the forward and feedback channels.
- Numerical simulations show that our optimal policy can reduce the age compared with other approaches.

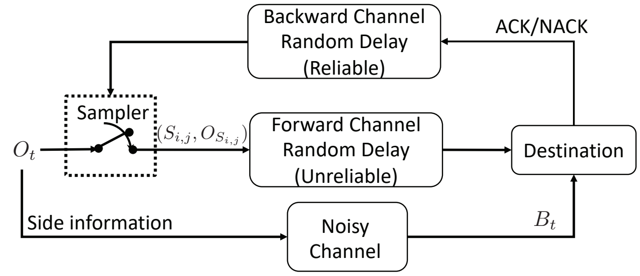

Figure 1: System model.

We remind that although our sampling problem is continuous time, it can be easily reduced to be discrete time. Therefore, the result of discrete time case is omitted in this paper.

II. Estimation and the AoI

A. System Model

Consider a status update system composed of the source, destination, source-to-destination channel, and destination-to-source channel, as is illustrated in Fig. 1. The source process O_t is sampled and delivered to the destination via the forward channel. The forward channel suffers from i.i.d. transmission failures, and α is the probability of failure. Upon the delivery, the destination then sends feedback about whether the transmission is successful (ACK) or unsuccessful (NACK). The feedback is sent via the feedback (backward) channel that is reliable with an i.i.d. random delay.

To clarify the system model, we set $i \in \{1, 2, \cdots\}$ as the label of a successful delivery in chronological order. Let us denote the ith *epoch* to be the time period between $i-1$th and ith successful deliveries. We denote M_i as the total number of samples attempted during the ith epoch. Then, M_i's are i.i.d. and has a geometric distribution with parameter $1 - \alpha$. We use j to describe sample process at the ith epoch, where we have $1 \leq j \leq M_i$. The case $j = 1$ implies that the previous forward transmission is successful. Upon delivery, the destination immediately sends the feedback to the sampler and arrives at $A_{i,j}$ via the backward channel with an i.i.d. delay $X_{i,j}$, which satisfies $\mathbb{E}[X_{i,j}] < \infty$. Then, the jth sample in the ith epoch is generated at $S_{i,j}$ and is delivered at $D_{i,j}$ through the forward channel with an i.i.d. delay $Y_{i,j}$, which satisfies $\mathbb{E}[Y_{i,j}] < \infty$.

We assume that the backward delays $X_{i,j}$'s and forward delays $Y_{i,j}$'s are mutually independent. In addition, the source generates a sample after receiving the feedback of the previous sample[1], i.e., $S_{i,j} \geq A_{i,j}$. In other words, we have a non-negative waiting time $Z_{i,j}$ for all epoch i and sample j. Thus, the forward channel is always available for transmission at

[1]This assumption arises from the stop-and-wait Automatic Repeat Request (ARQ) mechanism. When the backward delay $X_{i,j} = 0$, the policy that samples ahead of receiving feedback is always suboptimal. Because such a policy takes a new sample when the channel is busy and can be replaced by a policy that samples at exact time of receiving feedback [5]. When $X_{i,j} \neq 0$, however, it may be optimal to transmit before receiving acknowledgement, which is out of the scope of this paper.

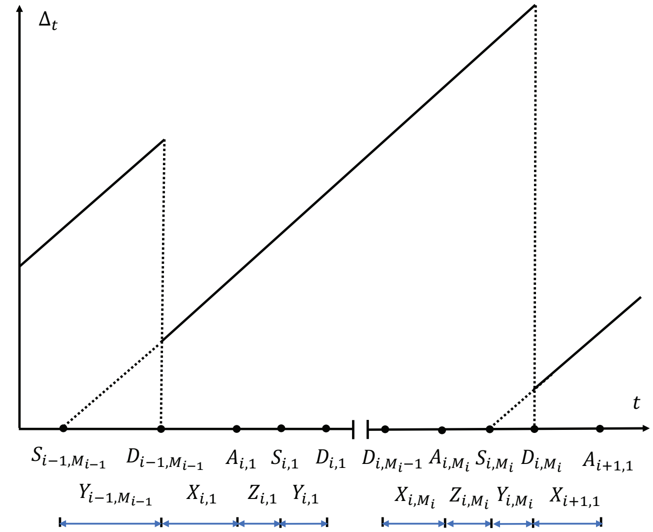

Figure 2: Evolution of the age Δ_t over time. The ith epoch starts from $D_{i-1,M_{i-1}}$ to D_{i,M_i}.

$S_{i,j}$. and we have $D_{i,j} = S_{i,j} + Y_{i,j}$. Finally, note that the total transmission delay needed in each epoch has a finite expectation by Wald's equation:

$$\mathbb{E}\left[\sum_{j=1}^{M_i}(X_{i,j}+Y_{i,j})\right] = \mathbb{E}\left[X_{i,j}+Y_{i,j}\right]\mathbb{E}\left[M_i\right] < \infty. \quad (1)$$

After introducing the system model, we describe age of information. Age of information (or simply *age*) is the metric for evaluating data freshness and is equal to the time elapsed between the current time t and the generation time of the freshest delivered packet [1]. Let $U_t = \max\{S_{i,M_i} : D_{i,M_i} \le t\}$. Note that only the M_ith sample is successfully delivered for the ith epoch. Then, the age of information Δ_t at current time t is defined as

$$\Delta_t = t - U_t. \quad (2)$$

We plot the evolution of the age (2) in Fig. 2. Upon each successful delivery time D_{i,M_i}, the age decreases to Y_{i,M_i}, the transmission delay of the newly generated packet. In other times, the age increases linearly over time. In all, the age is updated at the beginning of each epoch and keeps increasing during the epoch. Suppose that $D_{i,M_i} \le t < D_{i+1,M_{i+1}}$, then the age is also defined as

$$\Delta_t = t - S_{i,M_i}, \text{ if } D_{i,M_i} \le t < D_{i+1,M_{i+1}}. \quad (3)$$

B. Remote Estimation and Kalman Filter

To simplify, we introduce some notations. For any multi-dimensional vector O, we denote O^T as the transpose of O. We denote $I_{n\times n}, 0_{n\times m}$ as the $n\times n$ identity matrix and $n\times m$ zero matrix, respectively. For a given $n\times n$ matrix N, we set $tr(N)$ as the trace of N, i.e., the summation of the diagonal elements of N.

In this subsection, we assume that the destination is an estimator, and the source process O_t is an n-dimensional diffusion process that is defined as the solution to the following stochastic differential equation:

$$dO_t = -\Theta O_t dt + \Sigma dW_t, \quad (4)$$

where Θ and Σ are $n\times n$ matrices, and W_t is the n-dimensional Wiener process such that we have $\mathbb{E}[W_t W_s^T] = I_{n\times n}\min\{s,t\}$ for all $0 \le t,s \le \infty$. The process O_t represents the behavior of many physical systems [22].

The key difference from previous works, e.g., [6], [10], [15], [21] is that the estimator not only receives the accurate samples $O_{S_{i,j}}$ at time $S_{i,j}$ but also has an instant noisy observation of the process O_t, called B_t, as is illustrated in Fig. 1. The observation process B_t is an m-dimensional vector and is modeled as

$$B_t = HO_t + V_t, \quad (5)$$

where H is an $n\times m$ matrix and V_t is a zero mean white noise process such that for all $0 \le t,s \le \infty$, we have

$$\mathbb{E}[V_t V_s^T] = \begin{cases} R & t = s; \\ 0_{m\times m} & t \ne s, \end{cases} \quad (6)$$

where R is an $m\times m$ positive definite matrix. We suppose that W_t and V_t are uncorrelated such that we have $\mathbb{E}[W_t V_s^T] = 0_{n\times m}$ for all $0 \le t,s \le \infty$.

The objective of the estimator is to provide an estimation $\hat{O}_t$ for minimum mean squared error (MMSE) $\mathbb{E}[||O_t - \hat{O}_t||^2]$ based on the causally received information. Compared with [10], the MMSE in our study can be reduced due to the additional observation process B_t. Using the strong Markov property O_t [23, Eq. (4.3.27)] and the assumption that the sampling times are independent of O_t, as is shown our technical report [24], the MMSE estimator is determined by

$$\hat{O}_t = \mathbb{E}\left[O_t | \{B_\tau\}_{S_{i,M_i}\le\tau\le t}, O_{S_{i,M_i}}\right], t \in [D_{i,M_i}, D_{i+1,M_{i+1}}). \quad (7)$$

According to (7), we find that the MMSE estimator $\hat{O}_t$ is equal to that of the *Kalman filter* [22]. Therefore, in this work, we use the Kalman filter as the estimator. At time t, the Kalman filter utilizes both the exact sample $O_{S_{i,M_i}}$ and noisy observation B_t and provides the minimum mean squared error (MMSE) estimation $\hat{O}_t$. Let $N_t \triangleq \mathbb{E}[(O_t - \hat{O}_t)(O_t - \hat{O}_t)^T]$ be the covariance matrix of the estimation error $O_t - \hat{O}_t$. It is easy to see that the MMSE $\mathbb{E}[||O_t - \hat{O}_t||^2]$ of the Kalman filter at time t is equal to $tr(N_t)$.

The estimation process works as follows: Once a sample is delivered to the Kalman filter at time D_{i,M_i}, the Kalman filter re-initiates itself with the initial condition $N_t = 0_{n\times n}$ when $t = S_{i,M_i}$ and starts a new estimation session. Then, during the time period $[D_{i,M_i}, D_{i+1,M_{i+1}})$, the Kalman filter uses the observations B_t for $t \ge S_{i,M_i}$ over time to estimate the process O_t.

In this multiple dimension case, we have the following result:

Proposition 1. *For $t \in [D_{i,M_i}, D_{i+1,M_{i+1}})$ and $i = 0,1,2,\ldots$, the MMSE $tr(N_t)$ of the process O_t is an increasing function of the age Δ_t, where $\Delta_t = t - S_{i,M_i}$.*

Proof. See our technical report [24]. □

As a result of Proposition 1, when the sampling times $S_{i,j}$'s are independent of O_t, the MMSE is still an increasing function of age Δ_t. When $S_{i,j}$'s are related to O_t, the MMSE is not necessary a function of Δ_t.

Note that in one-dimensional case where $n = m = 1$, we use scalars $\theta, \sigma, h, r, n_t$ to replace the matrices $\Theta, \Sigma, H, R, N_t$, respectively. The Ornstein–Uhlenbeck (OU) process is defined as a one-dimensional special case of diffusion process (4) where $\theta > 0$ [25]. Then, we have

Proposition 2. *Suppose that $n = m = 1$. Then, for $t \in [D_{i,M_i}, D_{i+1,M_{i+1}})$ and $i = 0, 1, 2, \ldots$, the MMSE n_t of the OU process O_t is given by*

$$n_t = \bar{n} - \frac{1}{l + \left(\frac{1}{\bar{n}} - l\right) e^{2\sqrt{\theta^2 + \frac{\sigma^2 h^2}{r}}\Delta_t}}, \tag{8}$$

where $\Delta_t = t - S_{i,M_i}$ and the constants $\bar{n}, l$ are defined as

$$\bar{n} = \frac{-\theta r + \sqrt{(\theta r)^2 + \sigma^2 r h^2}}{h^2}, \tag{9}$$

$$l = \frac{h^2}{2\sqrt{(\theta r)^2 + \sigma^2 r h^2}}. \tag{10}$$

Moreover, from (8)*, n_t is a bounded and increasing function of the age Δ_t.*

Proof. See our technical report [24]. □

In the special case, when the side observation has zero knowledge of process O_t, i.e., $h = 0$ for $t \geq 0$, then the estimator $\hat{O}_t$ is equal to that in [10]. Therefore, Proposition 2 reduces to [10, Lemma 4], i.e., the MMSE n_t for $t \in [D_{i,M_i}, D_{i+1,M_{i+1}})$ and $i = 0, 1, 2, \ldots$ is given by

$$n_t = \frac{\sigma^2}{2\theta}\left(1 - e^{-2\theta\Delta_t}\right), \tag{11}$$

moreover, n_t for $h = 0$ is a bounded and increasing function of age Δ_t.

III. Problem Formulation for General Age Penalty

The function in Proposition 2 is not the only choice of nonlinear age functions. In this paper, to achieve data freshness in various applications, we consider a general type of age penalty function. The age penalty function $p : [0, \infty) \to \mathbb{R}$ is assumed to be non-decreasing and need not be continuous or convex. We further assume that $\mathbb{E}\left[\int_\delta^{\delta + \sum_{j=1}^{M_i}(X_{i,j}+Y_{i,j})} p(t)dt\right] < \infty$ and $\mathbb{E}\left[p\left(\delta + \sum_{j=1}^{M_i}(X_{i,j} + Y_{i,j})\right) dt\right] < \infty$ for any given δ.

We list another two categories of applications for age penalty functions. First, the age penalty functions describe the dissatisfactions of the stale information updates in various applications [4]: (i) the online learning in advertisement placement and online web ranking, e.g, power function $p(\delta) = \delta^a$ for $a > 0$; (ii) Periodic Inspection and Monitoring, e.g., $p(\delta) = \lfloor a\delta \rfloor$. Second, some applications are shown to be closely related to nonlinear age functions, such as autocorrelation function of the source, remote estimation, and information based data freshness metric [5].

We then define the sampling policies. We denote $\mathcal{H}_{i,j}$ as the sample path of the history information previous to $A_{i,j}$, including sampling times, forward channel conditions, and channels delays. We denote Π as the collection of sampling policies $\{S_{i,j}\}_{i,j}$, or equivalently the waiting times $\{Z_{i,j}\}_{i,j}$, such that $S_{i,j} \geq A_{i,j}$ for each (i,j), and $S_{i,j}(ds_{i,j}|\mathcal{H}_{i,j})$ is a *Borel measurable stochastic kernel* [26, Chapter 7] for any possible $\mathcal{H}_{i,j}$. Further, we assume that $T_i = S_{i,1} - S_{i-1,1}$ is a regenerative process: there exists $0 \leq k_1 < k_2 \ldots$ such that the post-k_j process $\{T_{k_j+i}, i = 0, 1 \cdots\}$ has the same distribution as the post-k_1 process $\{T_{k_1+i}, i = 0, 1 \cdots\}$ and is independent of the pre-k_j process $\{T_i, i = 1, 2, \cdots, k_j - 1\}$; in addition, $\mathbb{E}[k_{j+1} - k_j] < \infty$, $\mathbb{E}[S_{k_1,1}] < \infty$ and $0 < \mathbb{E}\left[S_{k_j+1,1} - S_{k_j,1}\right] < \infty$, $j = 1, 2, \cdots$ [2]

The authors in [15] have stated that: to reduce the estimation error related to the Wiener process, it may be optimal to wait on both the source and the destination before transmission. However, in this paper, it is sufficient to only wait at the source to minimize the age. To validate this statement, consider any policy that waits on both the source and the destination. We first remove the waiting time at the destination. The removed waiting time is then added to the following waiting time at the source. The replaced policy we propose has the same age performance as the former one.

Our objective in this paper is to optimize the long term average expected age penalty:

$$p_{\text{opt}} = \inf_{\pi \in \Pi} \limsup_{T \to \infty} \frac{1}{T}\mathbb{E}\left[\int_0^T p(\Delta_t)dt\right]. \tag{12}$$

A. Additional Assumption and its Rationale

We provide the following assumption:

Assumption 1. *(a) If $\alpha > 0$, the backward delay $X_{i,j} \in [0, \bar{x}]$, and the waiting time $Z_{i,j} \in [0, \bar{z}]$ for all i, j, where $\bar{z}$ satisfies*[3]

$$p(\bar{z}) \geq \frac{\mathbb{E}\left[\int_{Y_{i-1,M_{i-1}}}^{Y_{i-1,M_{i-1}} + X_{i,1} + Y'} p(t)dt\right]}{\mathbb{E}\left[X_{i,1} + Y'\right]}, \tag{13}$$

$$Y' \triangleq Y_{i,1} + \sum_{j=2}^{M_i}\left(X_{i,j} + Y_{i,j}\right). \tag{14}$$

(b) There exists an increasing positive function $v(\delta)$ such that the function $G(\delta) = \mathbb{E}\left[\int_\delta^{\delta+\bar{x}+\bar{z}+Y_{i,j}} |p(t)|dt\right]$ satisfies

[2] In this paper, we will optimize $\limsup_{T\to\infty}(1/T)\mathbb{E}\left[\int_0^T p(\Delta_t)dt\right]$. However, a nicer objective is optimizing $\lim_{n\to\infty}\mathbb{E}\left[\int_0^{D_{n,M_n}} p(\Delta_t)dt\right]$ $/\mathbb{E}\left[D_{n,M_n}\right]$. If T_i is a regenerative process, then the two objective functions are equal [27], [28]. If no conditions are applied, they are different.

[3] In this paper, we set the summation operator $\sum_{j=a}^{b}$ to be 0 if $b < a$ for any given integers a, b.

$\max_{\delta \geq 0} |G(\delta)/v(\delta)| < \infty$. *In addition, there exists* $\rho \in (0,1)$ *and positive integer* m*, such that*

$$\alpha^m \frac{\mathbb{E}\left[v(\delta + m\bar{x} + m\bar{z} + \sum_{j=1}^{m} Y_j)\right]}{v(\delta)} \leq \rho \qquad (15)$$

holds for all $\delta \geq 0$*, where* $Y_1, \cdots, Y_m$ *are an i.i.d. sequence with the same distribution as* $Y_{i,j}$*'s.*

When the forward channel is reliable, i.e., $\alpha = 0$, then Assumption 1 is negligible by letting $v(\delta) = G(\delta)$. Thus, Assumption 1 restricts on the choices of age penalty $p(\cdot)$ when $\alpha > 0$. Note that the optimal sampling policy of the cases $\alpha = 0$ and $X_{i,j} = 0$ has been solved in [4], [5].

In the following corollary, we provide a list of age penalties $p(\cdot)$ that Assumption 1 is satisfied for $\alpha > 0$.

Corollary 1. *For any one of the following conditions, Assumption 1 holds:*

(a) The penalty function $p(\cdot)$ *is bounded, i.e., there exists a constant* $\bar{p}$ *such that* $p(\delta) < \bar{p}$ *for all* $\delta \geq 0$*.*

(b) There exists $n > 0$ *such that* $p(\delta) = O(\delta^n)$[4] *and the* $Y_{i,j}$*'s have a finite* $n+1$*-momentum, i.e.,* $\mathbb{E}\left[Y_{i,j}^{n+1}\right] < \infty$*.*

(c) There exists $a > 0$ *and* $b < 1$ *such that* $\int p(\delta)d\delta = O(e^{a\delta^b})$ *and the* $Y_{i,j}$*'s are bounded.*

Proof. See our technical report [24]. □

In various fields of MDP, it is shown that the *value function* of an optimal policy is the solution to the *Bellman equation*. In this paper, we figure out a policy and its value function that is indeed the solution to the Bellman equation. If the Bellman equation has a unique solution, then our proposed policy is optimal. Otherwise, we cannot guarantee the optimality of our proposed policy. Assumption 1 arises from the contraction mapping assumption [29], [30] that guarantees the uniqueness. In other words, Assumption 1 serves as a sufficient condition to the uniqueness of the Bellman equation (but may not be necessary). Corollary 1 implies that there are a wide range of age penalty functions that satisfy Assumption 1. For example, the age penalty function derived in Proposition 2 satisfies Assumption 1. Indeed, Assumption 1 holds as long as the age penalty function grows slower than exponential only at some points.

The inequality (13) guarantees that the optimal policy we will provide in Section IV is feasible. There exists a constant $\bar{z}$ such that (13) holds; See our technical report [24] for the proof of this statement. The high-level idea of this statement is that the right hand side of (13) represents an average age penalty among the finite expected length and is no more than $p(\bar{z})$ for some $\bar{z}$. Note that the right hand side of (13) does not depend on i since the $Y_{i,j}$'s and the $X_{i,j}$'s are i.i.d and mutually independent.

[4]We denote $f(\delta) = O(g(\delta))$ if there exists constants c and δ' such that $|f(\delta)| \leq c|g(\delta)|$ for all $\delta > \delta'$.

IV. Optimal Sampling Policy

In this section, we provide an optimal solution to (12):

Theorem 1. *If* $p(\cdot)$ *is non-decreasing, the* $Y_{i,j}$*'s,* $X_{i,j}$*'s are i.i.d., mutually independent, and* $\mathbb{E}[Y_{i,j}] < \infty$*, and Assumption 1 follows, then the optimal solution to* (12) *for* $i = 1, 2, \ldots$ *is illustrated as follows:*

$$\begin{aligned} Z_{i,1}(\beta) = &\inf_z \Big\{ z \geq 0 : \\ &\mathbb{E}_{Y'}\left[p(Y_{i-1,M_{i-1}} + X_{i,1} + z + Y')\right] \geq \beta \Big\}, \quad (16) \\ Z_{i,j}(\beta) = &0 \quad j = 2, 3, \ldots, \quad (17) \end{aligned}$$

$Y' = Y_{i,1} + \sum_{j=2}^{M_i}(X_{i,j} + Y_{i,j})$*, and* β *is the unique solution to*

$$\begin{aligned} &\mathbb{E}\left[\int_{Y_{i-1,M_{i-1}}}^{Y_{i-1,M_{i-1}} + X_{i,1} + Z_{i,1}(\beta) + Y'} p(t)dt\right] \\ &- \beta\mathbb{E}\left[X_{i,1} + Z_{i,1}(\beta) + Y'\right] = 0. \quad (18) \end{aligned}$$

Moreover, $\beta = p_{opt}$ *is the optimal objective value of* (12)*.*

Proof. See Section V. □

In Theorem 1, the case $j = 1$ in (16) implies that the previous transmission is successful, and the system starts the new epoch from $i-1$ to i. Since the age drops to $Y_{i-1,M_{i-1}}$ at the successful delivery time $D_{i-1,M_{i-1}}$, the current age state at arrival time $A_{i,1}$ is $Y_{i-1,M_{i-1}} + X_{i,1}$. The case $j = 2, 3, \ldots$ in (17) implies that the previous transmission is instead unsuccessful and the system stays within epoch i.

Theorem 1 provides an optimal policy with an interesting structure. First, by (16), in each epoch, the optimal waiting time for the first sample $Z_{i,1}(\beta)$ has a simple threshold type structure on the current age $Y_{i-1,M_{i-1}} + X_{i,1}$. Since the waiting times for $j = 2, 3, \ldots$ are zero, Y' is the remaining transmission delay needed for the next successful delivery. Note that β is equal to the optimal objective value p_{opt} in problem (12). Therefore, the waiting time $Z_{i,1}(\beta)$ in (16) is chosen such that the expected age penalty upon delivery is no smaller than p_{opt}. Second, by (17), the source sends the packet as soon as it receives negative feedback, i.e., the previous transmission is not successful. This is quite different from most of the previous works assuming reliable channels, e.g., [4], [5], [8], [9], where for all samples, the source may wait for some time before transmissions.

In addition, the root of β in (18) can be solved efficiently. For simplicity, we set

$$f_1(\beta) = \mathbb{E}\left[\int_{Y_{i-1,M_{i-1}}}^{Y_{i-1,M_{i-1}} + X_{i,1} + Z_{i,1}(\beta) + Y'} p(t)dt\right], \quad (19)$$

$$f_2(\beta) = \mathbb{E}\left[X_{i,1} + Z_{i,1}(\beta) + Y'\right]. \quad (20)$$

Then, the function $f(\beta) \triangleq f_1(\beta) - \beta f_2(\beta)$ satisfies the following mathematical property:

Lemma 1. *(1)* $f(\beta)$ *is concave, and strictly decreasing in* $\beta \in [\underline{p}, \bar{p}] \cap \mathbb{R}$*, where* $\underline{p} = p(0)$ *and* $\bar{p} = \lim_{\delta \to \infty} p(\delta)$*.*

(2) There exists a unique root $\beta \in [\underline{p}, \bar{p}] \cap \mathbb{R}$ such that $f(\beta) = 0$.

Proof. See our technical report [24]. □

Algorithm 1: Bisection method for solving (18)

```
Given function f(β) = f1(β) − βf2(β). k1 close to p̲,
   k2 close to p̄, k1 < k2, and tolerance ϵ small.
repeat
    β = ½(k1 + k2)
    if f(β) < 0: k2 = β. else k1 = β
until k2 − k1 < ϵ
return β
```

As a result, we can use a low complexity algorithm such as bisection search and fixed-point iterations to obtain the optimal objective value p_{opt}. The bisection search approach to solving p_{opt} is illustrated in Algorithm 1.

One common sampling policy is the zero-wait policy, which samples the packet once it receives the feedback, i.e., $Z_{i,j} = 0$ for all (i, j) [3]. The zero-wait policy maximizes the throughput and minimizes the delay. However, by Theorem 1, the zero-wait policy may be suboptimal on age. The following result provides the necessary and sufficient condition when the zero-wait policy is optimal. We get

Corollary 2. *If $p(\cdot)$ is non-decreasing, the $Y_{i,j}$'s, $X_{i,j}$'s are i.i.d., mutually independent, and $\mathbb{E}[Y_{i,j}] < \infty$, and Assumption 1 follows, then the zero-wait policy is optimal if and only if*

$$\textit{ess}\inf \mathbb{E}_{Y'}\left[p(Y + X + Y')\right] \geq \frac{\mathbb{E}\left[\int_{Y}^{Y+X+Y'} p(t)dt\right]}{\mathbb{E}\left[X + Y'\right]}, \quad (21)$$

where $Y' = Y_{i,1} + \sum_{j=2}^{M_i}(X_{i,j} + Y_{i,j})$, $Y = Y_{i-1,M_{i-1}}, X = X_{i,1}$ and we denote $\textit{ess}\inf E = \inf\{e : \mathbb{P}(E \leq e) > 0\}$ for any random variable E.

Proof. See our technical report [24]. □

When the channel delays are constant, we can get from Corollary 2 that

Corollary 3. *If $p(\cdot)$ is non-decreasing and satisfies Assumption 1, and the $Y_{i,j}$'s, $X_{i,j}$'s are constants, then the zero-wait policy is the solution to problem* (12).

Proof. See our technical report [24]. □

Theorem 1 is an extension to some key results in [5], [9]. When the forward channel is reliable, i.e., $M_i = 1$ for all i or $\alpha = 0$, Theorem 1 can be reduced to the result in [9]. Further, when $M_i = 1$, we extends [9] in two folds: (i) The age penalty $p(\cdot)$ is allowed to be negative or discontinuous. (ii) The channel delays $Y_{i,1}, X_{i,1}$ have a finite expectation and do not need to be bounded. Note that when $M_i = 1$, Assumption 1 is negligible. When $M_i = 1$, and there is no backward delay ($X_{i,1} = 0$), our result reduces to [5, Theorem 1].

The study in [7, Theorem 2] proves the optimality of the zero-wait policy among the deterministic policies under unreliable forward channel. This result corresponds to Corollary 3, a special case of Theorem 1. Our paper extends [7] in two folds: (i) We allow the policy space Π to be randomized. Among randomized policies, due to the disturbances on the previous sampling times, the current sampling time is dependent on the previous ones, which is different from [7]. (ii) We consider random two-way delays, which extends the constant one-way delay in [7].

V. Proof of the Main Result

In this section, we provide the proof of Theorem 1. In Section V-A, we first show that the long term average problem is equivalent to a per-epoch problem. In Section V-B, we solve the per-epoch problem. In Section V-C, we summarize our technical contribution compared with related works. Due to the space limit, the detailed proof of this section is relegated to our technical report [24].

A. *Reformulation of Problem* (12)

In this subsection, we decompose the original problem to a per-epoch problem. The idea is motivated by recent studies that reformulate the average problem into a per-sample problem [4]–[6], [9], [10].

Recall that the age decreases to $Y_{i-1,M_{i-1}}$ at time $D_{i-1,M_{i-1}}$. Note that $Y_{i-1,M_{i-1}}$ is independent of the history information. Thus, the age evolution at the i^{th} epoch is independent of the sampling decisions from the previous epochs $0, 1, 2, ..., i-1$. Therefore, to minimize the average cost at each epoch separately is sufficient to minimize the average cost of long term multiple epochs.

Thus, for any epoch i, the original problem (12) satisfies

$$p_{\text{opt}} = \inf_{\pi \in \Pi_i} \frac{\mathbb{E}\left[\int_{Y_{i-1,M_{i-1}}}^{Y_{i-1,M_{i-1}} + \sum_{j=1}^{M_i}(X_{i,j} + Z_{i,j} + Y_{i,j})} p(t)dt\right]}{\mathbb{E}\left[\sum_{j=1}^{M_i}(X_{i,j} + Z_{i,j} + Y_{i,j})\right]}, \quad (22)$$

where we define the policy space Π_i as the collection of sampling decisions $(Z_{i,1}, Z_{i,2}, ...)$ at epoch i such that the stochastic kernel

$$Z_{i,j}(dz_{i,j} | y_{i-1,M_{i-1}}, x_{i,1}, z_{i,1}, y_{i,1}, \ldots, z_{i,j-1}, y_{i,j-1}, x_{i,j})$$

is Borel measurable. The difference between Π_i and Π is that the sampling decisions in Π_i do not depend on the history information from previous epochs (except $Y_{i-1,M_{i-1}}$). Hence, it is easy to find that $\Pi_i \subset \Pi$.

Using Dinkelbach's method [31], we can get

Lemma 2. *An optimal solution to* (22) *satisfies*

$$\inf_{\pi \in \Pi_i} \mathbb{E}\left[\int_{Y_{i-1,M_{i-1}}}^{Y_{i-1,M_{i-1}} + \sum_{j=1}^{M_i}(X_{i,j} + Z_{i,j} + Y_{i,j})} p(t)dt \right. \\ \left. - p_{opt} \sum_{j=1}^{M_i}(X_{i,j} + Z_{i,j} + Y_{i,j}) \;\middle|\; Y_{i-1,M_{i-1}}, X_{i,1}\right]. \quad (23)$$

Thus, for any epoch i, we will solve $Z_{i,1}, Z_{i,2}, ...$ according to (23).

B. Solution to Per-epoch Problem

We will solve problem (23) given that $Y_{i-1,M_{i-1}} = \delta$ and $X_{i,1} = x$. Since the epoch number i does not affect problem (23), in this subsection, we will remove the subscription i from $M_i, X_{i,j}, Y_{i,j}, Z_{i,j}$ and replace them by M, X_j, Y_j, Z_j for the ease of descriptions.

Different from [4]–[6], [10], the per-epoch problem (23) is an MDP and cannot be reduced to the per-sample problem in the sense that the age is not refreshed under failed transmissions. According to (23), we define the value function J_π under a policy $\pi \in \Pi_i$ with an initial age state $\delta \geq 0$ (at delivery time) and backward delay $x \geq 0$:

$$\begin{aligned} J_\pi(\delta, x) =& \mathbb{E}\Bigg[\int_\delta^{\delta + \sum_{j=1}^M (X_j + Z_j + Y_j)} p(t)dt \\ & - p_{\text{opt}} \sum_{j=1}^M (X_j + Z_j + Y_j) \ \Bigg| \ X_1 = x \Bigg] \quad (24) \\ =& \mathbb{E}\left[\sum_{j=1}^M g(\Delta_j, X_j, Z_j) \ \Bigg| \ \Delta_1 = \delta, X_1 = x \right], \quad (25) \end{aligned}$$

where the instant cost function $g(\delta, x, z)$ with state (δ, x) and action z is defined as

$$g(\delta, x, z) = \mathbb{E}_Y \left[\int_\delta^{\delta + x + z + Y} p(t)dt - p_{\text{opt}}(x + z + Y) \right], \quad (26)$$

where Y has the same delay distribution with the Y_j's and the age state evolution is described as

$$\Delta_{j+1} = \Delta_j + X_j + Z_j + Y_j, \quad j = 1, 2, ...M - 1, \quad (27)$$

with initial age state $\Delta_1 = \delta$ and initial backward delay x. Also, the policy $\pi \in \Pi_i$ has a Borel measurable stochastic kernel $Z_j(dz_j | \delta_1, x_1, z_1, \dots, \delta_j, x_j)$. Thus, $J_\pi(\delta, x)$ is Borel measurable and problem (23) is equivalent to a shortest path MDP problem. Solving (23) is equivalent to solving

$$J(\delta, x) = \inf_{\pi \in \Pi_i} J_\pi(\delta, x). \quad (28)$$

Besides, our shortest path problem (25) is equivalent to a discounted problem:

$$J_\pi(\delta, x) = \sum_{j=1}^\infty \alpha^{j-1} \mathbb{E}\left[g(\Delta_j, X_j, Z_j) \ \Big| \ \Delta_1 = \delta, X_1 = x \right]. \quad (29)$$

Note that (29) is motivated by [32, Chapter 5] that illustrates the discounted problem is equivalent to a special case of shortest path problem.

Recall that uncountable infimum of Borel measurable functions is not necessary Borel measurable. Problem (23) has an uncountable state space. Thus, the optimal value function $J(\delta, x)$ defined in (28) *may not be Borel measurable*[5]. One of the methods to overcome this challenge is to enlarge the policy spaces. We define a collection of policies Π'_i such that the stochastic kernel $Z_j(dZ_j | \delta_1, x_1, z_1, \dots, \delta_j, x_j)$ is universally measurable [26]. Note that every Borel measurable stochastic kernel is a universally measurable stochastic kernel, so we have $\Pi_i \subset \Pi'_i$.

[5]see [26], [29] for counterexamples. In discrete time case where the system time is slotted, we do not have this challenge.

Note that if $\pi \in \Pi'_i$, we denote $J_\pi(\delta, x)$ as the discounted cost of π given in (29). For all given age state δ and delay x, we define

$$J'(\delta, x) = \inf_{\pi \in \Pi'_i} J_\pi(\delta, x). \quad (30)$$

When $\alpha = 0$, problem (23) (equivalently, (28)) or the extended problem (30) becomes a single sample problem and there is no bound restriction to the instant cost function $g(\delta, x, z)$. However, in the unreliable transmission case where $\alpha > 0$, problem (23) contains multiple samples. In the case of multiple samples, most of the literatures of dynamic programming e.g., [26], [29], [30], [32]–[36] require that the instant cost function $g(\delta, x, z)$ is bounded from below. Here, we have

Lemma 3. *There exists a value λ such that $g(\delta, x, z) \geq -\lambda$ and $J_\pi(\delta, x) \geq -\lambda/(1 - \alpha)$ for all (δ, x, z) and any policy $\pi \in \Pi'_i$.*

By Lemma 3 and [26, Corollary 9.4.1], $J'(\delta, x)$ is lower semianalytic [26]. Note that any real-valued Borel measurable function is lower semianalytic. This allows us to consider the Bellman operator based on a general lower semianalytic function $u(\delta, x)$. For any deterministic and stationary policy $\pi \in \Pi_i$, we define an operator T_π on the function u with action $z \geq 0$:

$$\begin{aligned} & T_\pi u(\delta, x) \\ =& g(\delta, x, \pi(\delta, x)) + \alpha \mathbb{E}_{Y,X} \left[u(\delta + x + \pi(\delta, x) + Y, X) \right], \end{aligned} \quad (31)$$

where Y and X have the same distribution as the i.i.d. forward delay and backward delay, respectively. We also define the Bellman operator T on the function u:

$$Tu(\delta, x) = \inf_{z \geq 0} g(\delta, x, z) + \alpha \mathbb{E}_{Y,X} \left[u(\delta + x + z + Y, X) \right]. \quad (32)$$

Note that if the function $u(\delta, x)$ is Borel measurable, $Tu(\delta, x)$ is not necessary Borel measurable in the sense that uncountable infimum of Borel measurable functions is not necessary Borel measurable. However, if we extend $u(\delta, x)$ to be lower semianalytic, then $Tu(\delta, x)$ is also lower semianalytic [26, Chapter 7]. Thus, both T_π and T are well-defined. Note that the expectation on a lower semianalytic function has the same definition with the expectation on a Borel measurable function.

We denote $u_1 = u_2$ if $u_1(\delta, x) = u_2(\delta, x)$ for all $\delta, x \in [0, \infty)$. Using the notation T_π and T, the discounted problem (29) along with (30) has the following properties [26, Chapter 9.4]:

Lemma 4. *If $p(\cdot)$ is non-decreasing, the Y_j's, X_j's are i.i.d., mutually independent, $\mathbb{E}[Y_j] < \infty$, and $\mathbb{E}[X_j] < \infty$, then the optimal value function $J'(\delta, x)$ defined in* (30) *satisfies the Bellman equation:*

$$J' = TJ', \tag{33}$$

i.e., the optimal value function J' is a fixed point of T.

To derive an optimal policy, we first provide a measurable, stationary and deterministic policy called μ and then show that μ is the solution to problem (23).

Definition 1. *The stationary and deterministic policy $\mu(\delta, x)$ satisfies*

$$\mu(\delta, x) = \max\{b - \delta - x, 0\}, \tag{34}$$

$$b = \inf_{c} \left\{ c \geq 0 : \mathbb{E}\left[p(c + Y_1 + \sum_{j=2}^{M}(X_j + Y_j)) \right] \geq p_{opt} \right\}. \tag{35}$$

Equation (34) implies that $\mu \in \Pi_i$. Upon delivery of the first sample, age increases to $\Delta_2 = \delta + x + \mu(\delta, x) + Y_1$, which is larger than $\max\{\delta + x, b\}$. Then, the waiting time for the 2nd sample is $\mu(\Delta_2, X_2) = 0$. Thus, the waiting time at stage 2,... is 0 under μ. As a result, the policy μ is equivalent to (16) and (17) in Theorem 1. It remains to show that μ is indeed optimal to problem (23).

Recall that we denote $J_\mu(\delta, x)$ to be the value function with initial state δ, x under the policy μ. Then, we have the following key result:

Lemma 5. *If $p(\cdot)$ is non-decreasing, the Y_j's, X_j's are i.i.d., mutually independent, $\mathbb{E}[Y_j] < \infty$, and $\mathbb{E}[X_j] < \infty$, then the value function $J_\mu(\delta, x)$ satisfies*

$$J_\mu = TJ_\mu. \tag{36}$$

Lemma 5 tells that J_μ is a fixed point of T. From Lemma 4 (a), J' is also a fixed point of T. To show that $J' = J_\mu$, it remains to show that the fixed point of T is unique. If the age penalty $p(\cdot)$ is bounded, $J_\mu(\delta, x)$ is bounded. Then, according to the contraction mapping theorem, the bellman equation (33) has a unique bounded solution [28], [29], [36], i.e., $J_\mu = J'$. Note that there may be unbounded solutions to (33) [34], [35]. If $p(\cdot)$ is unbounded, we need sup-norm weighted contraction mapping assumption (Assumption 1 in our paper) to show that $J_\mu = J'$.

Let us denote $\Lambda = [0, \infty) \times [0, \bar{x}]$, where $\bar{x}$ is the bound of X_j in Assumption 1. The increasing function $v(\delta) : [0, \infty) \to \mathbb{R}^+$ is called the weighted function with $v(0) > 0$. Let $B(\Lambda)$ denote the set of all lower semianalytic functions $u : \Lambda \to \mathbb{R}$ such that $u(\delta, x)/v(\delta)$ is bounded in (δ, x). Note that any real-valued Borel measurable function is lower semianalytic. From [29, p. 47], [26, Lemma 7.30.2], $B(\Lambda)$ is complete under its norm. We have the following lemma:

Lemma 6. *If $p(\cdot)$ is non-decreasing, the Y_j's, X_j's are i.i.d., mutually independent, and $\mathbb{E}[Y_j] < \infty$, and Assumption 1 follows, then for all $\pi \in \Pi_i$, $J_\pi \in B(\Lambda)$.*

Under Assumption 1, the waiting time $z \in [0, \bar{z}]$. Hence, we will replace $\inf_{z \geq 0}$ by $\inf_{z \in [0, \bar{z}]}$ for the definition of T in (32). If a function u is lower semianalytic, then the function Tu is also lower semianalytic.

We have the following result:

Lemma 7. *If $p(\cdot)$ is non-decreasing, the Y_j's, X_j's are i.i.d., mutually independent, and $\mathbb{E}[Y_j] < \infty$, and Assumption 1 follows, the following conditions hold:*

(a) For any lower semianalytic function $u : \Lambda \to \mathbb{R}$, if $u \in B(\Lambda)$, then $T_\pi u \in B(\Lambda)$ for all deterministic and stationary policy $\pi \in \Pi_i$, and $Tu \in B(\Lambda)$.

(b) T has an m-stage contraction mapping with modulus ρ, i.e., for all $u_1, u_2 \in B(\Lambda)$,

$$\|T^m u_1 - T^m u_2\| \leq \rho \|u_1 - u_2\|, \tag{37}$$

where $\rho < 1$ as stated in Assumption 1, and the weighted sup-norm $\|\cdot\|$ is defined as:

$$\|u\| = \max_{(\delta,x)\in\Lambda} \frac{|u(\delta, x)|}{v(\delta)}. \tag{38}$$

(c) There exists a unique function $u \in B(\Lambda)$ such that $Tu = u$.

From Lemma 6, $J_\mu \in B(\Lambda)$. From Lemma 7(c), Lemma 5 and $J_\mu \in B(\Lambda)$, J_μ is the unique solution to $Tu = u$. From Lemma 4, $J_\mu = J'$ and μ is then the optimal policy in Π_i'.

Finally, since $\mu \in \Pi_i$ and $\Pi_i \subset \Pi_i'$, μ is also the optimal policy in Π_i and then $J = J_\mu$. Thus, μ is the solution to problem (23). Using the proof of Lemma 2, the expectation of $Y_{i-1,M_{i-1}}, X_{i,1}$ in (23) (while applying μ) equals 0. This immediately gets (18). Note that the definition of μ is the same as (16) and (17). Thus, we complete the proof of Theorem 1.

C. Discussion

Most of the existing studies on AoI sampling assume that the transmission channel is error-free, i.e., $M_i = 1$ for all i, e.g., [2], [4]–[6], [9], [10], [15]. Due to the renewal property, their original problems are reduced to a single sample problem. Similarly, our result is equivalent to a single epoch problem illustrated in (23). If $M_i = 1$, problem (23) reduces to a single sample problem, where there is only one decision $Z_{i,1}$ and is solved using convex optimization. However, when $M_i \neq 1$, problem (23) is an MDP that contains multiple samples. This MDP cannot be solved by convex optimization (e.g., [2], [4], [5], [9]) or optimal stopping rules (e.g., [6], [10], [15]) and leads to the curse of dimensionality.

Therefore, one of our technical contributions is to accurately solve the MDP in (23). We summarize the high-level idea of solving (23): First, in Lemma 4, we show the optimal policy among the extended policy space Π_i' with universally measurable stochastic kernel [26] satisfies the Bellman equation. Then, in Lemma 5, we provide an exact value function that is the solution to (23). Finally, under Assumption 1, Lemma 7 guarantees the uniqueness of the Bellman equation.

Finally, although we focus on continuous system time in this paper, our results can be easily reduced to the discrete time case by removing the content of measure theory.

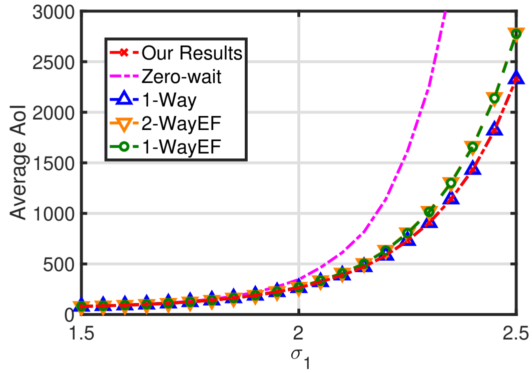

Figure 3: Average AoI versus the parameter σ_1 of the forward channel, where $\sigma_2 = 1.5$ and $\alpha = 0.8$.

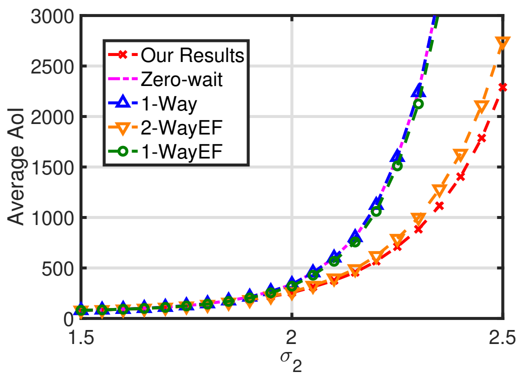

Figure 4: Average AoI versus the parameter σ_2 of the backward channel, where $\sigma_1 = 1.5$ and $\alpha = 0.8$.

VI. Numerical Results

In this section, we compare our optimal sampling policy with the following sampling policies:

1. Zero-wait: Let $Z_{i,j} = 0$, i.e., the source transmits a sample once it receives the feedback.

2. One-way (1-way): It falsely assumes that the backward delay $X_{i,j} = 0$ despite that $X_{i,j}$ may not be zero.

3. Two-way Error-free (2-wayEF) [9]: It assumes that the forward channel's probability of failure $\alpha = 0$ despite that α may not be zero.

4. One-way Error-free (1-wayEF) [5]: It assumes that $X_{i,j} = 0$ and $\alpha = 0$.

In this section, we consider linear age penalty $p(\delta) = 2\delta$ and lognormal distributions on both forward and backward delay with scale parameters σ_1, σ_2, respectively. Note that the lognormal random variable with scale parameter σ is expressed as $e^{\sigma R}$, where R is the standard normal random variable. The numerical results below show that our proposed policy always achieves the lowest average age.

Fig. 3 and Fig. 4 illustrate the relation between age and σ_1, σ_2, respectively. In Fig. 3, we plot the evolution of average age in σ_1 given that $\sigma_2 = 1.5$ and $\alpha = 0.8$. As σ_1 increases, the lognormal distribution of the forward channel becomes more heavy tailed. We observe that Zero-wait policy evolves much quicker than other policies in σ_1. In addition, 2-wayEF and 1-wayEF policies grow faster than the optimal policy in σ_1. In Fig. 4, we fix $\sigma_1 = 1.5$ and plot the average age of the listed policies in σ_2. Unlike Fig. 3, 1-way and 1-wayEF policies perform poorly, since they fails to take highly random backward delay into account.

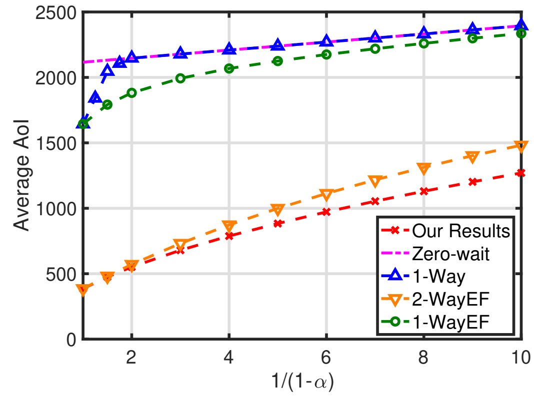

Figure 5: Average AoI versus $1/(1-\alpha)$, where $\sigma_1 = 1.5$ and $\sigma_2 = 2.3$.

Fig. 5 depicts the evolution of average age in $1/(1-\alpha)$, where $(\sigma_1, \sigma_2) = (1.5, 2.3)$, respectively. Note that $1/(1-\alpha)$ is the average number of samples attempted for a successful transmission. In Fig. 5, when $1/(1-\alpha)$ increases, the gap between 2-wayEF policy and our optimal policy increases. Also, 1-way and 1-wayEF fails to improve the age performance.

In summary, when either one of the channels is highly random, (i) Zero-wait policy is far from optimal, (ii) the age performance of 1-wayEF or 2-wayEF policy gets worse if the forward channel is more unreliable, (iii) 1-way and 1-wayEF polices are far from optimal if the backward channel is highly random.

VII. Conclusion

In this paper, we design a sampling policy to optimize data freshness, where the source generates the samples and sends to the remote destination via a fading forward channel, and the acknowledgements are sent back via a backward channel. We overcome the curse of dimensionality that arises from the time-varying forward channel conditions and the randomness of the channel delays in both directions. We reveal that the optimal sampling policy has a simple threshold based structure, and the optimal threshold is equal to the objective value of our problem and is computed efficiently.

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