Battle between Rate and Error in Minimizing Age of Information

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ABSTRACT
In this paper, we consider a status update system, in which update packets are sent to the destination via a wireless medium that allows for multiple rates, where a higher rate also naturally corresponds to a higher error probability. The data freshness is measured using age of information, which is defined as the age of the recent update at the destination. A packet that is transmitted with a higher rate, will encounter a shorter delay and a higher error probability. Thus, the choice of the transmission rate affects the age at the destination. In this paper, we design a low-complexity scheduler that selects between two different transmission rate and error probability pairs to be used at each transmission epoch. This problem can be cast as a Markov Decision Process. We show that there exists a threshold-type policy that is age-optimal. More importantly, we show that the objective function is quasi-convex or non-decreasing in the threshold, based on the system parameters values. This enables us to devise a low-complexity algorithm to minimize the age. These results reveal an interesting phenomenon: While choosing the rate with minimum mean delay is delay-optimal, this does not necessarily minimize the age.

CCS CONCEPTS
• Networks → Network performance evaluation; Network performance analysis;

KEYWORDS
Age of information, Markov decision process, Heterogenous transmission, Threshold-type policy

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1 INTRODUCTION
Age of information is a new metric that has attracted significant recent attention [3, 4, 10, 19]. This concept has been motivated by the rapid growth of real-time applications, e.g., health monitoring, automatic driving system, and agriculture automation, etc. For such applications, freshness of information updates is of utmost importance. However, traditional metric like delay cannot fully characterize the freshness of information updates. For example, if information is updated infrequently, then the updates are not fresh even though the delay is small. To this end, age of information, or simply the age, was proposed in [11] as a measure of the data freshness. Specifically, age of information is defined as the time elapsed since the generation of the most recently received status update.

There exist many works dealing with the age minimization problem. One class of works have focused on investigating optimal sampling and updating policies to minimize age of information. In [16], authors study the updating policy to minimize age in the presence of queuing delay. In [1, 7, 17, 20], sampling and updating polices are studied under energy constraint. In [1, 17], the authors assume that the channel is noiseless while in [20], authors assume that channel state is known a priori and updating cost is a function of channel state to ensure successful transmission. In [7], the authors consider transmission failure and investigate optimal sampling policy for age minimization under energy constraint. These works consider the effects of queueing delay, channel state, energy supply and minimize the age of information by controlling sampling and updating times, in which case they assume that there is only one transmission mode to transmit updates. However, in real systems, updates can be sent to a destination using heterogenous transmissions in terms of transmission delay and error probability. Two examples are provided as follows:

Error rate control: Error rate control scheme is managed at physical layer. In particular, the transmission rate is often adapted via modulation and coding scheme to meet a fixed target error rate [9]. It is known that choosing a lower target error rate corresponds to a lower transmission rate, and hence a longer transmission delay. On the other hand, a higher transmission rate (i.e., a shorter transmission delay) also corresponds to a higher transmission error probability of information delivery. Thus, there is a tradeoff between transmission delay and transmission error probability, both of which are affected by the target error rate.

Scheduling over channels in different frequencies: It is common that a device can access channels in different frequencies. For example, cellphones can access WiFi (high frequency) and LTE (low frequency). If updates are transmitted over such devices, then the age of information may experience different transmission properties based on the carrier frequency. In particular, it is known that it is hard for radio waves to distract obstacles that are in same or larger size than their wavelength. Thus, low-frequency radios (longer wavelength) are less vulnerable to blockage than high-frequency
radios, which implies that low frequency channels are more reliable than their high frequency counterparts. Of course, the higher frequency channels allow for higher rate (shorter delay) transmissions, resulting in a similar tradeoff between the transmission delay and transmission error probability.

The above examples clearly indicate that, transmission of updates can experience different transmission delays and error probabilities based on the choice of either target error rate or carrier frequency. In particular, a decrease in the transmission error probability will increase the chances of a successful update delivery (decrease age) while an increase in the transmission delay will increase the inter-delivery time (increase age). That is, the delay and error probability of a transmission mode affect the age in opposite directions. Thus, the key questions are: when is it optimal to use the lower transmission rate with a lower error probability?, which variable plays a more important role in determining the optimal actions?. To address these questions, we begin by investigating a status update system with two heterogeneous transmissions and obtain the optimal transmission selection policy to minimize the average age. Studying the two-rate scenario provides us with some insights in the optimal policy for a more general multi-rate (multi-error probability) scenario, which is discussed in Section 5, and provides basis for our future work. Specifically, our contributions are outlined as follows:

- We investigate the optimal trade-off between transmission delay and error probability for minimizing the age. We show that there exists a stationary deterministic optimal transmission selection policy. Moreover, we show that the optimal transmission selection policy is of threshold-type in terms of the age (Theorem 4.1). In particular, we show that the optimal action is a non-increasing (non-decreasing) function of the age if the mean delay of the low rate transmission is smaller (larger) than that of the high rate transmission. This result was not anticipated: For example, in [8, 12], it was shown that the optimal delay policy chooses the server with minimum mean delay whenever it is available. With this, one may expect that using the transmission with higher mean delay would worsen the age performance. Surprisingly, however, we show that choosing the transmission with higher mean delay can sometimes improve the age performance.

- We derive the average cost as a function of the threshold with the aid of the state transition diagram. We then optimize the threshold to minimize the average cost function. In particular, although the optimization problem is non-convex, we are able to show that if the mean delay of the low rate transmission is smaller than that of the high rate transmission, the objective function is quasi-convex; otherwise, the optimal policy chooses higher rate transmission (Theorem 4.6). This enables us to devise a low-complexity algorithm to obtain the optimal policy.

The remainder of this paper is organized as follows. The system model is introduced in Section 2. In Section 3, we map the problem to an equivalent problem which can be regarded as an average cost MDP, and then formulate the MDP problem. In Section 4, we explore the structure of the optimal policy and properties of average cost function, and devise an efficient algorithm. In Section 5, we provide a discussion on multi-rate scenario. In Section 6, we provide numerical results to verify our theoretical results.

2 SYSTEM MODEL

We consider a status update system, in which update packets are sent to the destination via a wireless medium with varying transmission delay and error probability. The update packets are generated whenever the wireless medium becomes idle. We assume that there are two heterogeneous transmissions available for updating, namely low rate and high rate transmissions. The high rate transmission offers a shorter transmission delay than low rate transmission; while low rate transmission offers more reliable transmission than high rate transmission. A decision maker chooses a transmission mode for each transmission opportunity. We denote the set of transmission modes as \( U \neq \{1, 2\} \), where 1 and 2 denote the low rate and high rate transmissions, respectively. We use \( P = \{p_j : 0 < p_j < 1, j \in U\} \) and \( D = \{d_j : 0 < d_j < \infty, j \in U\} \) to denote the set of transmission error probabilities and transmission delays, respectively. Transmission \( j \in U \) corresponds to transmission delay \( d_j \) and transmission error probability \( p_j \). We assume that \( d_1 > d_2 \) and \( p_1 < p_2 \).

We use \( Y_i \) to denote the transmission delay of packet \( i \), where \( Y_i \in D \). Let \( D_i \) denote the delivery time of packet \( i \). Since updates are generated whenever the wireless medium becomes idle, \( D_i \) equals to the generation time of packet \( i + 1 \). Also, we have \( D_i = \sum_{j=1}^i Y_j \). At any time \( t \), the most recently received update packet is generated at time

\[
U(t) = \max \{ D_i : D_{i+1} \leq t \}. \tag{1}
\]

Then, the age of information, or simply the age is defined as

\[
\Delta(t) = t - U(t). \tag{2}
\]

The age \( \Delta(t) \) is a stochastic process that increases with \( t \) between update packets and is reset to a smaller value upon the successful delivery of a fresher packet. We suppose that the age \( \Delta(t) \) is right-continuous. As shown in Fig. 2, packet 2 is sent at time \( D_1 \) and its
delivery time is $D_2 = D_1 + Y_2$. Since this packet transmission fails, the age does not reset to a smaller value at $D_2$. Packet 3 transmission starts at $D_2$, which is successfully delivered at time $D_3$. Thus, the age increases linearly until it reaches to $\Delta(D_3) = Y_1 + Y_2 + Y_3$ before packet 3 is successfully sent, and then drops to $\Delta(D_3) = Y_3$ at $D_3$.

3 OPTIMIZATION PROBLEM

We use $u_i$ to denote which transmission mode (low rate or high rate) is selected to transmit packet $i$, where $u_i \in \mathcal{U}$. In particular, if $u_i = 1$ (or $u_i = 2$), then packet $i$ is transmitted using the low (or high) rate transmission, encounters transmission delay $d_1$ (or $d_2$), and is received successfully with probability $1 - p_1$ (or $1 - p_2$). A transmission selection policy $\pi$ specifies a transmission selection decision for each stage. For any policy $\pi$, we define the total average age as

$$\bar{\Lambda}(\pi) = \limsup_{n \to \infty} \frac{\mathbb{E}[\int_0^{D_n} \Delta(t)dt]}{\mathbb{E}[D_n]}.$$  \hspace{1cm} (3)

Our goal is to seek a transmission selection policy that solves the total average age minimization problem as follows:

$$\bar{\Lambda}^\ast = \min_{\pi \in \Pi} \bar{\Lambda}(\pi),$$  \hspace{1cm} (4)

where $\bar{\Lambda}^\ast$ denotes the optimal total average age. Let $\Pi$ denote the set of all causal transmission selection policies, in which the policy $\pi \in \Pi$ depends on the history and current system state.

3.1 Equivalent Mapping of Problem (4)

We decompose the area under the curve $\Delta(t)$ into a sum of disjoint geometric parts as shown in Fig. 2. Observing the area in interval $[0, D_n]$, the area can be regarded as the concatenation of the areas $Q_i$. Then,

$$\int_0^{D_n} \Delta(t)dt = \sum_{i=0}^{n-1} [Q_i].$$  \hspace{1cm} (5)

Let $a_i$ denote the age at time $D_i$, i.e., $a_i = \Delta(D_i)$. Then, $Q_i$ can be expressed as

$$Q_i = a_i Y_{i+1} + \frac{1}{2} Y_{i+1}^2.$$  \hspace{1cm} (6)

Recall that $D_n = \sum_{i=0}^{n-1} Y_{i+1}$. With Eq. (5) and Eq. (6), the total average age is expressed as

$$\limsup_{n \to \infty} \frac{\sum_{i=0}^{n-1} \mathbb{E}[a_i Y_{i+1} + \frac{1}{2} Y_{i+1}^2]}{\sum_{i=0}^{n-1} \mathbb{E}[Y_{i+1}]}.$$  \hspace{1cm} (7)

With this, the optimal transmission selection problem for minimizing the total average age can be formulated as

$$\bar{\Lambda}^\ast = \min_{\pi \in \Pi} \limsup_{n \to \infty} \frac{\sum_{i=0}^{n-1} \mathbb{E}[a_i Y_{i+1} + \frac{1}{2} Y_{i+1}^2]}{\sum_{i=0}^{n-1} \mathbb{E}[Y_{i+1}]}.$$  \hspace{1cm} (8)

The problem is hard to solve in current form. Thus, we provide an equivalent mapping for it. A problem with parameter $\beta$ is defined as follows:

$$p(\beta) = \min_{\pi \in \Pi} \limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{E}[(a_i - \beta) Y_{i+1} + \frac{1}{2} Y_{i+1}^2].$$  \hspace{1cm} (9)

Lemma 3.1. The following statements are true:

(i) $\Lambda^\ast \geq \beta$ if and only if $p(\beta) \geq 0$;

(ii) If $p(\beta) = 0$, then the optimal transmission selection policies that solve (8) and (9) are identical.

Proof. The proof is similar to that of Lemma 3.5 in [2]. The difference is that we use the boundedness of transmission delay while in [2], the boundedness of inter-sampling time is used. The detailed proof is provided in our technical report [18]. \hfill \square

By Lemma 3.1, if $\beta = \Lambda^\ast$, then the optimal transmission selection policies that solve (8) and (9) are identical. With this, given $\beta$, we formulate the problem (9) as an infinite horizon average cost per stage MDP in Section 3.2 and show that the optimal policy for (9) is of threshold-type in Section 4.1. Since the value of $\beta$ is arbitrary, we will be able to conclude that the optimal policy for (8) is of threshold-type. In addition, in Section 4.2, we are able to devise a low-complexity algorithm to obtain the optimal threshold.

3.2 The MDP problem of (9)

From [5], given $\beta$, problem (9) is equivalent to an average cost per stage MDP problem. The components of the MDP problem are described as follows:

- **States:** The system state at stage $i$ is the age $a_i$. In this paper, we consider the state space $S = \{a = ld_i + vd_2 : l, v \in \{0, 1, \ldots \}\}$. If the initial state is outside $S$, then eventually the state will enter $S$ with state $d_1$ or $d_2$; otherwise, a successful packet transmission never occurs. In fact, the maximal probability that no transmission succeeds after $l$ stages is $p^l_{d_2}$, which decreases with number of stages $l$. After state enters $S$, it will stay in $S$ onwards (since transmission delay is either $d_1$ or $d_2$). Note that $S$ is unbounded since successful packet transmission happens with certain probabilities.

- **Actions:** At delivery time $D_{i-1}$, the action that is chosen for stage $i$ is $u_i \in \mathcal{U}$. The action $u_i$ determines the transmission delay. For example, if $u_i = 1$, then the transmission delay at stage $i$ is $d_1$.

- **Transition probabilities:** Given the current state $a_i$ and action $u_i$ at stage $i$, the transition probability to the state $a_{i+1}$ at the stage $i + 1$ is defined as

$$P(a_{i+1} = a' | a_i = a, u_i = u) = \begin{cases} p_u & \text{if } a' = a + d_u, \\ 1 - p_u & \text{if } a' = a + d_{u'}, \\ 0 & \text{otherwise.} \end{cases}$$  \hspace{1cm} (10)

- **Costs:** Given state $a_i$ and action $u_i$ at stage $i$, the cost at the stage is defined as

$$C(a_i, u_i) = (a_i - \beta) d_u + \frac{1}{2} d_u^2.$$  \hspace{1cm} (11)

Given $\beta$, the average cost per stage under a transmission selection policy $\pi$ is given by

$$J(\pi, \beta) = \limsup_{n \to \infty} \frac{1}{n} \mathbb{E}_{\pi} \left[ \sum_{i=0}^{n-1} C(a_i, u_i) \right].$$  \hspace{1cm} (12)

Our objective is to find a transmission selection policy $\pi \in \Pi$ that minimizes the average cost per stage, which can be formulated as
Problem 1 (Average cost MDP)

\[
\min_{\pi \in \Pi} J(\pi, \beta).
\]  

(13)

We say that a transmission selection policy \( \pi \) is average-optimal if it solves the problem in (13). A transmission selection policy is a sequence of decision rules, i.e., \( \pi = (\xi_1, \xi_2, \cdots) \), where a decision rule \( \xi_i \) maps the history of states and actions, and the current state to an action. A transmission selection policy is called a stationary deterministic policy if \( u_i = \xi(a_i) \) for all \( i \in \mathbb{N}^+ \), where \( \xi : S \rightarrow U \) is a deterministic function. Stationary deterministic policies are the easiest to be implemented and evaluated. However, there may not exist a stationary deterministic policy that is average-optimal [5]. Next, we show that there exists a stationary deterministic transmission selection policy that is average-optimal. Moreover, we show that the optimal policy is of threshold-type.

4 STRUCTURE OF AVERAGE-OPTIMAL POLICY AND ALGORITHM DESIGN

In this section, we investigate the structure of the average-optimal policy and propose an efficient algorithm for the original problem (8).

4.1 Threshold Structure of Average-Optimal Policy

4.1.1 Threshold structure: The following theorem states that there exists a threshold-type stationary deterministic policy that is average-optimal. In particular, the problem is divided into two cases based on the relation between \( d_1(1 - p_2) \) and \( d_2(1 - p_1) \). Under these two cases, the threshold-type average-optimal policy shows opposite behaviors.

Theorem 4.1. There exists a stationary deterministic average-optimal transmission selection policy that is of threshold-type. Specifically,

(i) If \( d_1(1 - p_2) \leq d_2(1 - p_1) \), then the average-optimal policy is of the form \( \pi^* = (\zeta^*, \xi^*, \cdots) \), where

\[
\zeta^*(a) = \begin{cases} 
2 & \text{if } 0 \leq a \leq a_1', \\
1 & \text{if } a_1' < a,
\end{cases}
\]

(14)

where \( a_1' \) denotes the age threshold.

(ii) If \( d_1(1 - p_2) \geq d_2(1 - p_1) \), then the average-optimal policy is of the form \( \pi^* = (\zeta^*, \xi^*, \cdots) \), where

\[
\zeta^*(a) = \begin{cases} 
1 & \text{if } 0 \leq a \leq a_2', \\
2 & \text{if } a_2' < a,
\end{cases}
\]

(15)

where \( a_2' \) denotes the age threshold.

Proof. Please see Section 4.1.2.

Define the mean delay of transmission mode \( j \in U \) as

\[
d_j = \frac{d_j}{1 - p_j}.
\]

(16)

By Theorem 4.1 (i), when the age exceeds a certain threshold, the optimal policy chooses the transmission with smaller mean delay. This result reveals an interesting phenomenon: While the transmission with minimum mean delay is the optimal decision for minimizing the average delay, this does not necessarily minimize the age. In particular, when the age is below a certain threshold, the average age is reduced by choosing a faster transmission that has a higher mean delay (i.e., a higher error probability). The reason is that if successful, the age remains low. If it fails, it provides an opportunity to generate a later packet that can be transmitted in a shorter period of time. In Section 4.2, based on Theorem 4.1 (ii), we will show that the average-optimal policy under \( d_1(1 - p_2) \geq d_2(1 - p_1) \) is to choose high rate transmission for each transmission opportunity. This is reasonable because both the delay and mean delay (including the impact of the error probability) of high rate transmission is shorter than that of low rate transmission.

4.1.2 Proof of Theorem 4.1. One way to investigate the average cost MDPs is to relate them to the discounted cost MDPs. To prove Theorem 4.1, we (i) address a discounted cost MDP, i.e., establish the existence of a stationary deterministic policy that solves the MDP and then study the structure of the optimal policy; and (ii) extend the results to the average cost MDP problem in (13).

Given an initial state \( a \), the total expected \( \alpha \)-discounted cost under a transmission selection policy \( \pi \in \Pi \) is given by

\[
V^\alpha(a; \pi) = \lim_{n \to \infty} \mathbb{E}\left[ \sum_{i=0}^{n-1} \alpha^i C(a_i, u_i) \right],
\]

(17)

where \( 0 < \alpha < 1 \) is the discount factor. Then, the optimization problem of minimizing the total expected \( \alpha \)-discounted cost can be cast as

Problem 2 (Discounted cost MDP)

\[
V^\alpha(a) \equiv \min_{\pi \in \Pi} V^\alpha(a; \pi),
\]

(18)

where \( V^\alpha(a) \) denotes the optimal total expected \( \alpha \)-discounted cost. A transmission selection policy is said to be \( \alpha \)-discounted cost optimal if it solves the problem in (18). In Proposition 4.2, we show that there exists a stationary deterministic transmission selection policy which is \( \alpha \)-discounted cost optimal and provide a way to explore the property of the optimal policy.

Proposition 4.2. (a) The optimal total expected \( \alpha \)-discounted cost \( V^\alpha \) satisfies the following optimality equation:

\[
V^\alpha(a) = \min_{u \in U} Q^\alpha(a, u),
\]

(19)

where

\[
Q^\alpha(a, u) = C(a, u) + \alpha p_u V^\alpha(a + d_u) + \alpha(1 - p_u) V^\alpha(d_u).
\]

(20)

(b) The stationary deterministic policy determined by the right-hand-side of (19) is \( \alpha \)-discounted cost optimal.

(c) Let \( V_n^\alpha(a) \) be the cost-to-go function such that \( V_n^\alpha(a) = \frac{d_1 - d_2}{p_1 - p_2} a \) and for \( n \geq 0 \)

\[
V_{n+1}^\alpha(a) = \min_{u \in U} Q_{n+1}^\alpha(a, u),
\]

(21)

where

\[
Q_{n+1}^\alpha(a, u) = C(a, u) + \alpha p_u V_n^\alpha(a + d_u) + \alpha(1 - p_u) V_n^\alpha(d_u).
\]

(22)

Then, we have that for each \( a \), \( V_n^\alpha(a) \to V^\alpha(a) \) as \( n \to \infty \).

Proof. See our technical report [18].

Next, with the optimality equation (19) and value iteration (21), we show that the optimal policy is of threshold-type in Lemma 4.3.

**Lemma 4.3.** Given a discount factor $\alpha$,
(i) if $(1 - \alpha p_2) d_1 \leq (1 - \alpha p_1) d_2$, then the $\alpha$-discounted cost optimal policy is of threshold-type, i.e., the optimal action is a non-increasing function of the age.
(ii) if $(1 - \alpha p_2) d_1 \geq (1 - \alpha p_1) d_2$, then the $\alpha$-discounted cost optimal policy is of threshold-type, i.e., the optimal action is a non-decreasing function of the age.

**Proof.** Please see Appendix A

This lemma proves that the $\alpha$-discounted cost optimal policy is of threshold-type. Next, we extend the results to average cost MDP and show that there exists a stationary deterministic average-optimal policy which is of threshold-type. Based on the results in [15], we have the following lemma, which provides a candidate for average-optimal policy.

**Lemma 4.4.** (i) Let $\alpha_n$ be any sequence of discount factors converging to 1 with $\alpha_n$-discounted cost optimal stationary deterministic policy $\pi^{\alpha_n}$. There exists a subsequence $\alpha_m$ and a stationary policy $\pi^*$ that is a limit point of $\pi^{\alpha_n}$.
(ii) If $d_1(1 - p_2) \leq d_2(1 - p_1)$, $\pi^*$ is of threshold-type in (14); if $d_1(1 - p_2) \geq d_2(1 - p_1)$, $\pi^*$ is of threshold-type in (15).

**Proof.** See our technical report [18].

By [15], under certain conditions (A proof of these conditions verification is provided in our technical report [18]), $\pi^*$ is average-optimal.

### 4.2 Algorithm Design

Recall that if $p(\beta) = 0$, then the optimal transmission selection policies that solve (8) and (9) are identical. Given $\beta$, the optimal policy that solves (9) is of threshold-type by Theorem 4.1 and then (9) can be re-expressed as

$$p(\beta) = \begin{cases} 
\min_{\pi \in \Pi_1} J(\pi, \beta), & \text{if } d_1(1 - p_2) < d_2(1 - p_1) \quad (23) \\
\min_{\pi \in \Pi_2} J(\pi, \beta), & \text{if } d_1(1 - p_2) \geq d_2(1 - p_1) \quad (24) 
\end{cases}$$

where $\Pi_1$ and $\Pi_2$ denote the sets of threshold-type policies in (14) and (15), respectively. Thus, the optimal policy that solves (8) can be obtained with two steps:

- **Step (i):** For each $\beta$, find the $\beta$-associated average-optimal policy $\pi^{\beta}_u$ such that $p(\beta) = J(\pi^{\beta}_u, \beta)$.
- **Step (ii):** Find $\beta^*$ such that $p(\beta^*) = 0$. This implies $\pi^{\beta^*}_u$ solves (8).

To narrow our searching range in (ii), in Lemma 4.5, we provide a lower bound $\beta_{\min}$ and an upper bound $\beta_{\max}$ of $\beta^*$. Then, for (i), we only need to pay attention to $p(\beta)$ for $\beta \in [\beta_{\min}, \beta_{\max}]$.

In particular, within the range of $\beta$, we show that $J$ in (23) is quasi-convex in a threshold related variable, which enables us to devise a low-complexity algorithm based on golden section search. Moreover, we show that $\pi^{\beta}_{\beta^*}$ that solves (24) always chooses $u = 2$, which allows us to get the optimal policy for (8) directly.

**Lemma 4.5.** The parameter $\beta^*$ is lower bounded by $\beta_{\min} \leq 1.5d_1$ and upper bounded by $\beta_{\max} \leq \min \left\{ \frac{1}{\epsilon_1 p_1} + 0.5d_1, \frac{1}{\epsilon_2 p_1} + 0.5d_2 \right\}$.

**Proof.** See our technical report [18].

In the following content, we provide a theoretical analysis step by step for our algorithm design in Algorithm 1, which returns the optimal threshold and optimal average age for (8).

**Algorithm 1:** Threshold-based Age-Optimal Policy

```plaintext
1. given $d_1, d_2, p_1, p_2, \tau = (\sqrt{5} - 1)/2$, tolerance $\epsilon_1, \epsilon_2, l = \beta_{\min}, r = \beta_{\max}$.
2. while $r - l > \epsilon_1$
do
3. $\beta = \frac{r + l}{2}$.
4. if $d_1(1 - p_2) \geq d_2(1 - p_1)$ then
5. $m = 0, k = 0, \tau' = J(0, 0, \beta)$;
6. else
7. $k_{\max} = \lfloor \frac{\log_\tau \tau' + 1}{\log_\tau \beta} \rfloor$.
8. foreach $k \in \{0, 1, \ldots, k_{\max}\}$ do
9. if $J(\tau, k_1, \beta) > J(\tau, k_2, \beta)$ then
10. $y_1 = y_2 = y_1 = y_1 = (y_1 - y_2) \tau$.
11. while $p_1 y_1 \geq y_2$ and $y_1 > y_2 > \epsilon_2$ do
12. if $J(y_1, k_1, \beta) > J(y_1, k_2, \beta)$ then
13. $y_2 = y_2 - (y_2 - y_1) \tau$.
14. else
15. $y_1 = y_2 = y_2$.
16. end
17. end
18. end
19. $l = \lfloor \log_\tau y_1 \rfloor, c_{l_2} = \lfloor \log_\tau y_2 \rfloor$.
20. $m = \arg \min_{u \in \{1, 2\}} J(\tau, p_1, \beta, \beta)$.
21. end
22. if $J(\tau, p_1, \beta, \beta) \leq J'$ then
23. $l' = J(\tau, p_1, \beta, \beta)$.
24. end
25. if $J' \geq 0$ then
26. $l = \beta$;
27. else
28. $r = \beta$.
29. end
30. end
31. end
32. end
33. end
34. end
```

**Step (i):** Find the optimal policy $\pi^{\beta^*}_u$: Note that both of the threshold-type policies defined in (14) and (15) result in a Markov chain with a single positive recurrent class. Thus, given threshold $a_1^*$ in (14) or $a_2^*$ in (15), we can obtain the expression of average cost under the corresponding threshold-type policy with aid of state transition diagram. With this, we obtain some nice properties of average cost function in Theorem 4.6, which enables us to get a low-complexity algorithm. Before providing the result, we define the integer threshold which will be used in the theorem and algorithm.

Recall that age $a \in S$ is expressed as the sum of multiple $d_1$’s and $d_2$’s. Note that under the threshold-type policy in (14), if $a \leq a_1^*$, $\zeta(a) = 2$. This implies that if $a \leq a_1^*$, $a$ is in the form $a = d_1 + ld_2, j \in \mathcal{U}, l \in \mathbb{N}$. Thus, it is sufficient to use the following integer threshold to represent the threshold-type policy in (14).

$$m_1 \pm \min \left\{ l : d_1 + ld_2 > a_1^*, l \in \mathbb{N} \right\}, \quad (25)$$

$$n_1 \pm \min \left\{ l : d_2 + ld_2 > a_1^*, l \in \mathbb{N} \right\}. \quad (26)$$
Table 1: Notations

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<th>Description</th>
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<tbody>
<tr>
<td>$A_1 = p_{11}^1 (d_1(t) - d_1(t-p_1)) (d_1(t) + 1) - d_1(t-p_1)$</td>
<td></td>
</tr>
<tr>
<td>$B_1 = - A_2 + p_{21}^1 (1 - p_1) (0.5d_1^2 - \beta d_1) \left(1 + \frac{1}{p_1}\right) + (1 - p_1) (0.5d_2^2 - \beta d_2) + \frac{2}{p_1}$</td>
<td></td>
</tr>
<tr>
<td>$C_1 = (1 - p_1) (0.5d_2^2 - \beta d_2 + 1 + 1/p_1) (d_2(t) + 1)$</td>
<td></td>
</tr>
<tr>
<td>$D_1 = (d_1(t-p_1) - d_1(t-p_1)^2) d_1 p_1^1$</td>
<td></td>
</tr>
<tr>
<td>$A_2 = p_{12}^1 (d_2(t-p_2) - d_2(t-p_1)) (d_2(t-k_2) - d_2(t-k_1))$</td>
<td></td>
</tr>
<tr>
<td>$B_2 = - p_{21}^1 (1 - p_1) (0.5d_2^2 - \beta d_2) + d_2^2 + \frac{2}{p_1}$</td>
<td></td>
</tr>
<tr>
<td>$- (d_2(t-p_2) - d_2(t-p_1)) (d_2(t-k_2) - d_2(t-k_1))$</td>
<td></td>
</tr>
<tr>
<td>$C_2 = (1 - p_1) (0.5d_2^2 - \beta d_2 + \frac{2}{p_1})$</td>
<td></td>
</tr>
<tr>
<td>$D_2 = (d_2(t-p_1) - d_2(t-p_1)^2) d_1$</td>
<td></td>
</tr>
</tbody>
</table>

With this, the threshold policy in (14) is rewritten as

$$
\zeta^*(a) = \begin{cases} 
2 & \text{if } a < d_1 + m_1 d_2 \text{ and } a < d_2 + n_1 d_2, \\
1 & \text{if } a \geq d_1 + m_1 d_2 \text{ or } a \geq d_2 + n_1 d_2.
\end{cases}
$$

(27)

Similarly, the policy in (15) can be rewritten as

$$
\zeta^*(a) = \begin{cases} 
1 & \text{if } a < d_1 + m_2 d_1 \text{ and } a < d_2 + n_2 d_1, \\
2 & \text{if } a \geq d_1 + m_2 d_1 \text{ or } a \geq d_2 + n_2 d_1.
\end{cases}
$$

(28)

where

$$
m_2 = \min \{1 : d_1 + l d_2 > a_2^*, l \in \mathbb{N}\},
$$

$$
n_2 = \min \{1 : d_2 + l d_1 > a_2^*, l \in \mathbb{N}\}.
$$

(29)

(30)

Based on the analysis, (23) and (24) can be re-expressed as

$$
\rho(\beta) = \begin{cases} 
\min_{m_1,k_1} J_1(m_1,k_1,\beta) & \text{if } d_1(t-p_2) < d_2(t-p_1), \\
\min_{m_2,k_2} J_2(m_2,k_2,\beta) & \text{if } d_1(t-p_1) \geq d_2(t-p_1).
\end{cases}
$$

(31)

(32)

where $k_1 = n_1 - m_1$ and $k_2 = n_2 - m_2$. Also, we use $J_1(m_1,k_1,\beta)$ and $J_2(m_2,k_2,\beta)$ to denote the average cost under the policies in (27) and (28), respectively. We have $k_2 \in \mathcal{K}_2 \subseteq \{0,1\}$ and $k_1 \in \mathcal{K}_1 \subseteq \{0, \ldots, \lceil d_1/p_1 \rceil\}$. In particular, according to the definition of $m_2$ and $n_2$, we have $(m_2 + 1) d_1 > a_2^* \geq d_2 + (n_2 - 1) d_1$ and $d_2 + n_2 d_1 > a_2^* \geq m_2 d_2$. Substitute $k_2 = n_2 - m_2$ into these two inequalities, we get $k_2 \in \mathcal{K}_2$. Similarly, we have $k_1 \in \mathcal{K}_1$.

In Theorem 4.6, we provide some nice properties for $J_1$ and $J_2$, which enables us to develop a low-complexity algorithm. To this end, we make a change of variable in Theorem 4.6 (i), i.e., $m_1$ is replaced with $\log_{p_2}(y)$ in $J_1$, where $y \in (0,1]$. Some notations used in this theorem are defined in Table 1.

Theorem 4.6. Given $\beta \in [\beta_{min}, \beta_{max}]$,

(i) if $d_1(t-p_2) < d_2(t-p_1)$, then the average cost is given by

$$
J_1(y,k_1,\beta) = \frac{A_1 y^2 + B_1 y + C_1 + D_1 y \log_{p_2}(y)}{1 - p_1 + (-1 + p_1 + p_2^1 (1 - p_2^1) y},
$$

(33)

where $y \equiv p_{21}^1$. Moreover, $J_1(y,k_1,\beta)$ is quasi-convex in $y$ for $0 < y \leq 1$, given $k_1 \in \mathcal{K}_1$.

(ii) if $d_1(t-p_2) \geq d_2(t-p_1)$, then optimal average cost is given by

$$
J_2(0,0,\beta) = \frac{A_2 + B_2 + C_2}{1 - p_1}.
$$

(34)

Moreover, the average-optimal policy chooses $u = 2$ at every transmission opportunity.

Proof. Proof of part (i) is provided in Appendix B. For part (ii), the key idea is to show that for all $m_2$, $J_2(m_2,k_2,\beta)$ is non-decreasing in $k_2 \in \mathcal{K}_2$, and then $J_2(m_2,0,\beta)$ is non-decreasing in $m_2$. This implies that the optimal decision is $u = 2$ at each transmission opportunity. Due to the space limitation, detailed proof is provided in our technical report [18].

With the property in Theorem 4.6 (i), we are able to use golden section search [14] to find the optimal value of $y$ under condition $d_1(t-p_2) < d_2(t-p_1)$. The details are provided in Algorithm 1 (Line 12-20). Note that $\log_{p_2}(\cdot)$ is one-to-one functions. Thus, after obtaining the optimal $y$ that minimizes $J_1$, the corresponding optimal threshold policy $m_1$ can be easily obtained by comparing $J_1(p_2^1 \log_{p_2}(y), k_1, \beta)$ and $J_2(0, \log_{p_2}(y), k_1, \beta)$. The details are provided in Algorithm 1 (Line 21-22). Till now, we have solved $\min_{m_1} J_1(m_1, k_1, \beta)$ given $k_1$ and $\beta$. Note that $k_1 \in \mathcal{K}_1$ has finite and countable values. Then, we can easily solve (31) under condition $d_1(t-p_2) < d_2(t-p_1)$ by searching for the optimal $k_1$ in the finite set $\mathcal{K}_1$.

Note that Theorem 4.6 (ii) applies to all $\beta \in [\beta_{min}, \beta_{max}]$ including $\beta^*$. Thus, the optimal policy for (8) is to take $u = 2$ at every transmission opportunity, which is returned directly in Algorithm 1 (Line 4-5). Under this condition, the step (ii) is only used to find optimal average cost $\beta^*$ for (8).

Moreover, in Line 9-10, we add a judgement sentence, i.e., if the condition $\frac{\partial J_1(u, k_1, \beta)}{\partial y} |_{y = 1} < 0$ is satisfied, then we can directly obtain the optimal $m_1$ without running golden section method. This further reduces the algorithm complexity. The judgement is based on the fact that $\lim_{y \to 0} \frac{\partial J_1(u, k_1, \beta)}{\partial y} |_{y = 1} < 0$ (this is proved in the proof of Theorem 4.6 in Appendix B) and $J_1$ is quasi-convex. Thus, if $\frac{\partial J_1(u, k_1, \beta)}{\partial y} |_{y = 1} < 0$, then $J_1$ is non-increasing in $y$.

Step (ii): Find $\beta^*$: By Lemma 3.1, if $\rho(\beta) > 0$, then $\beta < \beta^*$; if $\rho(\beta) < 0$, then $\beta > \beta^*$. Thus, we can use bisection method to search for $\beta^*$. The details are provided in Algorithm 1 (Line 2-3 and Line 29-33).

5 A DISCUSSION ON THE GENERAL MULTI-RATE TRANSMISSION SELECTION PROBLEM

For the multi-rate transmission selection (from more than two-rate) problem, we assume that there are $N \in \mathbb{N}^+$ transmission modes for selection such that transmission delays and error probabilities satisfy $d_j > d_{j+1}$ and $p_j < p_{j+1}$, for $j \in \{1, 2, \ldots, N-1\}$, respectively. Since the transmission delays and transmission error probabilities affect the age in opposite direction, it is difficult to determine the optimal policy. Thanks to the results obtained for the two-rate transmission selection problem, we obtain some useful insights for the general multi-rate transmission selection.
Table 2: Optimal threshold versus delay

<table>
<thead>
<tr>
<th>$d_2=1$</th>
<th>$d_1=1.5d_2$</th>
<th>$d_1=1.7d_2$</th>
<th>$d_1=1.9d_2$</th>
<th>$d_1=2.1d_2$</th>
<th>$d_1=2.3d_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0.0)</td>
<td>(0.0)</td>
<td>(1.2)</td>
<td>(3.4)</td>
<td>(15,16)</td>
<td></td>
</tr>
<tr>
<td>(0.1)</td>
<td>(0.1)</td>
<td>(1.2)</td>
<td>(3.4)</td>
<td>(15,16)</td>
<td></td>
</tr>
<tr>
<td>(0.1)</td>
<td>(0.1)</td>
<td>(1.2)</td>
<td>(3.4)</td>
<td>(15,16)</td>
<td></td>
</tr>
</tbody>
</table>

In particular, for the two-rate transmission selection problem, the optimal action is a non-increasing function of age under condition $d_1 \leq d_2$ and a non-decreasing function of age under condition $d_1 \geq d_2$, where $d_1$ and $d_2$ are mean delays of low and high rate transmissions, respectively, as defined in (16). With this, we can infer that for $N > 2$, the optimal policy will have following properties:

- (i) if $d_1 \leq d_2 \leq \cdots \leq d_N$, then the optimal action will be a non-increasing function of the age.
- (ii) if $d_1 \geq d_2 \geq \cdots \geq d_N$, then the optimal action will be a non-decreasing function of the age.

For the cases that are not covered in (i) and (ii), we need more investigation on the property of the optimal policy. In addition, if we can show that the optimal policy for the general multi-rate problem is of threshold-type, then machine learning algorithms can be used to determine the optimal threshold. For example, if we regard each threshold-type policy with a certain threshold as a bandit, then basic bandit algorithms like UCB can be exploited to find the optimal threshold [13].

6 NUMERICAL RESULTS

In this section, we present some numerical results to explore the performance of the threshold-based age-optimal policy and verify our theoretical results.

First, we consider an update system, in which $p_1 = 0.4$ and $p_2 = 0.75$. Table 2 illustrates the relation between the optimal threshold versus the transmission delay $d_2$ and the delay ratio $d_1/d_2$ under condition $d_2(1-p_1) > d_1(1-p_2)$. The threshold $(m_1, n_1)$ in the table is obtained by Algorithm 1. We observe that the threshold increases with either $d_2$ or the delay ratio $d_1/d_2$. Note that when the age is below the threshold, transmission mode 2 is selected. This observation implies that transmission mode 2 becomes more preferable either when $d_2$ increases with fixed $d_1$ or when $d_1/d_2$ increases with fixed $d_2$.

In Fig. 3, we consider an update system, in which the transmission delays are $d_1 = 10$ and $d_2 = 8$. We use “Delay-Optimal” to denote the optimal policy that minimizes the average updating delay by always choosing the transmission mode with minimum mean delay [8, 12]. Moreover, we use “Age-Optimal” to denote the optimal policy that is obtained from Algorithm 1. We use “Random” to denote the policy that chooses $u = 1$ with probability $p$. We compare our threshold-based “Age-Optimal” policy with “Random” policies and the “Delay-Optimal” policy, where $p \in \{0.25, 0.5\}$. Fig. 3a (3b) illustrates the total average age in (7) versus transmission error probability $p_1 (p_2)$ given $p_2 = 0.5 (p_1 = 0.5)$. The dashed line in the figure marks the point at which $d_1(1-p_2) = d_2(1-p_1)$. The left and right (right and left) of the line corresponds to conditions $d_1(1-p_1) < d_2(1-p_1)$ and $d_1(1-p_2) > d_2(1-p_2)$ in Fig. 3a (3b), respectively. As we can observe, the age-optimal policy outperforms other plotted policies. This agrees with Theorem 4.1. Moreover, the results confirm that the delay-optimal policy does not necessarily minimize the age. In particular, the gap between delay-optimal and age-optimal policy becomes larger as $p_1 (p_2)$ approaches to the left side (right side) of the dashed line in Fig. 3a (Fig. 3b). The jump in the curve of the delay-optimal policy is incurred by the switch between two transmission modes. For example, in Fig. 3a, delay-optimal policy chooses $u = 1$ on the left side of the dashed line, while chooses $u = 2$ on the right side. This is because transmission mode 1 has smaller mean delay on the left side while transmission mode 2 has smaller mean delay on the right side.

7 CONCLUSION

In this paper, we studied the transmission selection problem for minimizing age of information in information update system with heterogeneous transmissions. We assume that there are two different transmissions with varying delay and error probability. We showed that there exists a stationary deterministic optimal transmission selection policy which is of threshold-type in age (Theorem 4.1). This result reveals an interesting phenomenon: If the mean delay of the low rate transmission is smaller than that of high rate transmission, then the optimal action chooses the one with higher mean delay when age is smaller than a certain threshold. This is in contrary with the delay-optimal policy that always chooses the transmission with lower mean delay. In addition, we showed that if the mean delay of the low rate transmission is smaller than that of high rate transmission, the average cost is quasi-convex in a threshold related variable; otherwise, the optimal policy chooses $u = 2$ for each transmission opportunity (Theorem 4.6). This enabled us to design a low-complexity algorithm to obtain the optimal policy (Algorithm 1). For the future work, we plan to study the multi-rate scenario with more than two selections of heterogeneous transmissions based on the insights discussed in Section 5.

ACKNOWLEDGMENTS

This work was funded in part through NSF grants: CNS-1901057, CNS-2007231, CNS-1618520, and CNS-1409336, and an Office of Naval Research under Grant N00014-17-1-241.
By Proposition 4.2, we only need to show that for $n \in \mathbb{N}$,

$$Q_{n+1}^{\alpha}(a_1, 1) - Q_{n+1}^{\alpha}(a_2, 1) \geq Q_{n+1}^{\alpha}(a_2, 2) - Q_{n+1}^{\alpha}(a_2, 2),$$

which is true by Corollary 4.1. This completes the proof.

**REFERENCES**


Hence, substitute (42) and (44) into (40) and we obtain
\[
\frac{p_0 d_1}{1 - p_0} = \sum_{l=0}^{m_1+k_1} x_1 + \sum_{l=0}^{m_1+k_2} x_2 + \ldots + \sum_{l=0}^{m_1+k_t} x_t = \frac{1}{1 - p_0}.
\]
Substituting (58)-(61) into \(\sum_{l=0}^{m_1+k_1} x_1 + \sum_{l=0}^{m_1+k_2} x_2 + \ldots + \sum_{l=0}^{m_1+k_t} x_t = 1\), we obtain \(x_0\) as
\[
x_0 = \frac{1}{1 - p_1} \left(1 - p_2 \right) \left(1 - p_1 \right) \frac{1}{1 - p_2}.
\]

The average cost \(J_1(m_1, k_1, \beta)\) is expressed as
\[
J_1(m_1, k_1, \beta) = \sum_{l=0}^{m_1+k_1-1} C(d_2 + l d_2, 2) x_1 + \sum_{l=0}^{m_1+k_2-1} C(d_2 + l d_2, 2) x_2 + \ldots + \sum_{l=0}^{m_1+k_t-1} C(d_2 + l d_2, 2) x_t.
\]

The proof is similar to part (i).

**B PROOF OF THEOREM 4.6 (I)**

We first obtain expression of average cost in (33) with aid of state transition diagram. The state transition diagram under the policy in (14) is given in Fig. 4. Define the state steady probabilities \(x_1, x'_1, z_1, z'_1\) under policy in (14) as
\[
\begin{align*}
    x_1 & \pm \mathbb{P}(a = d_2 + ld_2), & 0 \leq l & \leq m_1 + k_1, \\
    x'_1 & \pm \mathbb{P}(a = d_2 + (m_1 + k_1)d_2 + ld_1), & l & \geq 0, \\
    z_1 & \pm \mathbb{P}(a = d_1 + ld_2), & 0 \leq l & \leq m_1, \\
    z'_1 & \pm \mathbb{P}(a = d_1 + m_1 d_2 + ld_1), & l & \geq 0.
\end{align*}
\]

Based on the state transition diagram, balance equations can be obtained as follows:
\[
\begin{align*}
x_0 &= (1 - p_2) \left( \sum_{l=0}^{m_1-1} z_l + \sum_{l=0}^{m_1+k_1-1} x_1 \right), \\
p_2 x_2 &= x_1 + 1, & l & \in \{0, \ldots, m_1 \}, \\
p_2 z_1 &= z_1 + 1, & l & \in \{0, \ldots, m_1 \}, \\
p_1 x_2 &= x'_1 + 1, & l & \in \{0, \ldots, m_1 \}, \\
p_1 z_1 &= z'_1 + 1, & l & \in \{0, \ldots, m_1 \}.
\end{align*}
\]
Solving the equations (53)-(57), we obtain the expressions of \(x_1, x'_1, z_1, z'_1\) in terms of \(x_0\) as follows:
\[
\begin{align*}
x_1 &= p_2^l x_0, & l & \in \{0, \ldots, m_1 \}, \\
z_1 &= \frac{p_2^{m_1+k_1} x_0}{1 - p_2^{m_1+k_1}}, & l & \in \{0, \ldots, m_1 \}.
\end{align*}
\]
where \( W = d_2(1-p_1) - d_1(1-p_2) \). By condition \( d_2(1-p_1) > d_1(1-p_2) \), \( W > 0 \). Note that \( r \leq -1 + p_1 + (1 - p_2) = p_1 - p_2 < 0 \) since \( 0 < p_1 < p_2 < 1 \). Thus, \( H(y) \geq r + 1 - p_1 = p_2^k(1-p_2) > 0 \), for \( 0 < y \leq 1 \). Hence, \( \frac{\partial h(y,k_1,\beta)}{\partial y} \) is positive (negative) if and only if \( G(y) \) is positive (negative). Next, we will show our final result by analyzing two different cases.

**Case 1:** If \( d_2(k_1 + 1) - d_1 \geq 0 \), then \( G(y) \geq -\frac{k_1}{\ln p_2} > 0 \), for \( 0 < y \leq 1 \). In this case, \( \frac{\partial h(y,k_1,\beta)}{\partial y} > 0 \), for \( 0 < y \leq 1 \). Thus, \( h(y,k_1,\beta) \) is increasing in \( y \), for \( 0 < y \leq 1 \). Note that since \( D_1 < 0 \) (by condition \( d_2(1-p_1) > d_1(1-p_2) \)) and \( p_2 < 1 \), \( \lim_{y \to 0} h(y,k_1,\beta) = -C_1r + B_1 + D_1p_2 + \lim_{y \to 0} \frac{\partial h(y,k_1,\beta)}{\partial y} = (1-p_2) < 0 \). Thus, if \( h(1,k_1,\beta) \leq 0 \), then B2 holds; if \( h(1,k_1,\beta) > 0 \), then B3 holds.

**Case 2:** If \( d_2(k_1 + 1) - d_1 < 0 \), then \( G(y) \) is decreasing in \( y \) and
\[
y_1 = \frac{d_1}{d_2(k_1+1)-d_1}\ln p_2 > 0 \]
is a turning point such that \( G(y) > 0 \) when \( y < y_1 \) and \( G(y) < 0 \) when \( y > y_1 \).

If \( y_1 \geq 1 \), then \( G(y) \geq 0 \) for \( 0 < y \leq 1 \). Thus, \( \frac{\partial h(y,k_1,\beta)}{\partial y} \geq 0 \), which implies that \( h(k_1,\beta) \) is increasing in \( y \) for \( 0 < y \leq 1 \). Hence, B2 or B3 holds as explained in case 1.

If \( y_1 < 1 \), then \( G(y) > 0 \) when \( y < y_1 \) and \( G(y) < 0 \) when \( 1 > y > y_1 \), which implies \( h(k_1,\beta) \) first increases and then decreases for \( 0 < y \leq 1 \). We claim that if \( y_1 < 1 \), then \( h(1,k_1,\beta) \geq 0 \) for \( \beta \in [\beta_{\min},\beta_{\max}] \) and \( k_1 \in K_1 \). Recall that \( \lim_{y \to 0} h(y,k_1,\beta) < 0 \). With this, the claim implies that \( h(y,k_1,\beta) \) starts with some negative value and increases to zero at some point \( y' \) and after \( h(y,k_1,\beta) \) becomes positive, it will keep positive for \( y' < y \leq 1 \) (B3 holds). It remains to show that our claim holds.

Since \( y_1 = \frac{d_1}{d_2(k_1+1)-d_1}\ln p_2 < 1 \), we have
\[
d_1 > d_2(k_1 + 1) - d_1 = \frac{1}{2\ln p_2} \]

After some algebraic manipulation and simplification, \( h(1,k_1,\beta) \) is expressed as
\[
h(1,k_1,\beta) = p_2^{k_1} W \left( d_2(k_1 + 1) - d_1 - \frac{d_2}{\ln p_2} (1 - p_2) \right) + p_2^{k_1} (1-p_2)(1-p_1) \left( \frac{1}{1-p_1} + 0.5 \right) d_2^2 - \left( \frac{1}{1-p_2} + 0.5 \right) d_2^2 + p_2^{k_1} (d_2 - d_1) (1-p_1) (1-p_2) \beta \]

Since \( p_2^{k_1} (d_2 - d_1) (1-p_1) (1-p_2) < 0 \), \( h(1,k_1,\beta) \) is decreasing in \( \beta \) and hence \( h(1,k_1,\beta) \geq h(1,k_1,\beta_{\max}) \geq h(1,k_1,\beta_{\min}) + \frac{0.5}{\ln p_2} d_2^2 \). It remains to show that \( h(1,k_1,\beta_{\min}) + \frac{0.5}{\ln p_2} d_2^2 \geq 0 \). It is equivalent to show that \( \frac{1}{\beta_{\min}} \frac{1}{d_2^2} d_2^{k_1} + 0.5 d_2^2 \geq 0 \) since \( d_2^2 \geq 0 \). Let \( z = \frac{d_2}{d_2^2} \), substitute this into \( \frac{1}{\beta_{\min}} \frac{1}{d_2^2} d_2^{k_1} + 0.5 d_2^2 \) and obtain a function of \( z \) denoted by \( w(z) \). By (67), the first derivative of \( w(z) \) satisfies
\[
w'(z) = 2p_2^{k_1} (1-p_2)^2 z + p_2^{k_1} (1-p_2) \left( \frac{1}{1-p_1} + 0.5 \right)(1-p_1) + p_2^{k_1} (1-p_2) \left( \frac{1}{1-p_2} + 0.5 \right)(1-p_1) \]

The second inequality in (71) holds since \( 0 < p_1 < p_2 < 1 \). Thus, \( w(z) \) is increasing and we have
\[
w(z) \geq w(k_1 + 1) = \frac{1}{2\ln p_2} \]

where
\[
y'(p_1,p_2,k_1) = (p_1 - p_2)(1-p_2)p_2^{k_1} + 2(p_1 - p_2)\ln p_2 + 2k_1(1-0.5p_2)\ln p_2 + (1-p_2)0.5p_2 - 1 \]

To show \( w(z) \geq 0 \), we only need to show \( y'(p_1,p_2,k_1) \leq 0 \). Actually,
\[
\theta y'(p_1,p_2,k_1) = (p_1 - p_2)(1-p_2)p_2^{k_1} \ln p_2 + k_1(2-p_1)\frac{1}{p_2} - 1 - \ln p_2 \]

where the inequality holds since \( \theta(p_1,p_2,k_1) \) decreases with \( p_2 \) and \( \theta(p_1,1,k_1) = 0.5p_1 \). Actually,
\[
\frac{\partial \theta(p_1,p_2,k_1)}{\partial p_2} = (2p_2 - 1 - p_1)p_2^{k_1} \ln p_2 + 2p_2^{k_1} - 2p_2^{-1} - \frac{2}{p_2} \]

< \( (p_2 - 1)p_2^{k_1} \ln p_2 + 2p_2^{k_1} - 2p_2^{-1} - \frac{2}{p_2} \)

\[
\leq -(p_2 - 1)p_2^{k_1} \frac{1}{2p_2} + 2p_2^{k_1} - 2p_2^{-1} - \frac{2}{p_2} \]

< \( -(p_2 - 1)\frac{1}{2p_2} + 2p_2^{k_1} - 2p_2^{-1} - \frac{2}{p_2} \)

< \( 0 \)

where (76) holds since \( p_2 - p_1 > 0 \) and \( \ln p_2 < 0 \); (77) holds by \( s(x) \leq \frac{1}{x} + 2\ln x \geq 0 \) for \( 0 < x \leq 1 \); (78) holds since \( p_2^{k_1} \leq 1 \); (79) holds since \( p_2 < 1 \). In particular, the first derivative of \( s(x) \) is \( s'(x) = -\frac{1}{x^2} + \frac{2}{x} \). Thus, \( s(x) \) decreases when \( x \leq 0.5 \) (since \( s'(x) \leq 0 \) when \( x \leq 0.5 \)) and then increases when \( x \geq 0.5 \). Thus, \( s(x) \) increases when \( x \geq 0.5 \). By (74), \( y \) increases with \( p_2 \) and \( y(p_1,p_2,k_1) \leq y(p_1,1,k_1) = 0 \). This completes our proof.