Supplementary Material

Proof of Lemma 4 for Class IV Trees

Lemma 4. If we have \( x \prec y \), then for all \( j \in V \), we also have \( x + e_j \prec y + e_j \).

Proof: Let \( u = x + e_j \), and \( z = y + e_j \). Since \( x \prec y \), \( s(y) - s(x) = k \geq 0 \). Thus, \( s(z) - s(u) = k \).

We need to show that \( \forall i = 1, 2, \ldots, s(x) + 1 \)

\[
t^u_i \leq t^z_i + K \tag{1}
\]

We now consider the following cases. Recall that policy \( \pi_{IV} \) schedules the network with the new arrival as well.

Case 1: Packet in state \( u \) will reach the sink before the newly arrived packet, and packet \( i+k \) in state \( z \) will also reach the sink before the newly arrived packet. Further, neither \( i \) nor \( i+k \) is the newly arrived packet (see Figure 1(a)).

In this case, \( t^u_i = t^z_i + d^u \) and \( t^u_{i+k} = t^z_{i+k} \). Therefore, for all such packets, (1) holds.

Case 2: Packet in state \( u \) is either the newly arrived packet or will reach the sink only after the newly arrived packet reaches, and packet \( i+k \) in state \( z \) will reach the sink before the newly arrived packet and is not the newly arrived packet (see Figure 1(b)).

In this case, the \( i \)th packet in state \( x \) becomes the \( (i+1) \)th packet in state \( u \). We show that \( t^u_i \leq t^x_i \) as follows. It is easy to see that \( d^u_i \leq d^x_i \). Therefore, by Lemma 2, \( t^u_i \leq t^x_i \) if and only if \( t^u_{i-1} \leq t^x_{i-1} \). Iteratively substitute \( i \) by \( i-1 \) until packet \( i \) is a packet that will reach the sink before the newly arrived packet. For this packet, from Case 1, we know that \( t^u_i = t^z_i \). Hence, it follows that \( t^u_i \leq t^z_i \) for all packets \( i \) that satisfy the condition in Case 2.

For packets in state \( z \), the situation is the same as in Case 1. Therefore, it follows that \( t^u_i \leq t^z_i \leq t^z_{i+k} = t^z_{i+k} \).

Case 3: Packet in state \( u \) is either the newly arrived packet or will reach the sink before the newly arrived packet reaches, and packet \( i+k \) in state \( z \) is either the newly arrived packet or will reach the sink after the newly arrived packet reaches (Figure 2).

We prove (1) by contradiction. Suppose that for some \( i \), \( t^u_i > t^z_{i+k} \).

Suppose that the newly arrived packet is in one of the first \( K \) nodes in the equivalent linear network, say, at node \( d \leq K \) (Figure 2(a)). Then \( t^u_i = t^z_{i-1} + d^u_i \), and \( t^z_{i+k} \geq t^z_{i+k-1} + d \). Hence, \( t^u_i > t^z_{i+k} \) implies that \( t^z_{i-1} > t^z_{i+k-1} + d - d^u_i = t^z_{i+k} \). By iteratively substituting \( i \) by \( i-1 \), we either obtain \( i = 1 \) in state \( u \) or \( i+k \) is a packet that reaches the sink before the newly arrived packet according to state \( z \). If \( i = 1 \), \( t^u_i = t^z_0 = d \leq t^z_{i+k} \), and thus we get a contradiction. If \( i+k \) is a packet that reaches the sink before the newly arrived packet, then from Case 1, we get a contradiction. Hence, (1) must hold.

Suppose that the newly arrived packet arrives at a node \( > K \) in the equivalent linear network, say, at node \( d > K \) (Figure 2(b)). Suppose that for some \( i \), \( t^u_i > t^z_{i+k} \). If \( i \) is a packet that lies in one of the first \( K \) nodes in state \( u \), using the same argument as above, we get a contradiction. Otherwise, if \( t^u_i = t^z_0 \leq d \), since \( t^z_{i+k} \geq d \), we get a contradiction. If \( t^u_i = t^z_{i-1} + K + 1 > t^z_{i+k} \), this implies that \( t^z_{i-1} > t^z_{i+k-1} + K + 1 \). Hence, iteratively substituting \( i \) by \( i-1 \), and arguing as above, we again get a contradiction.

Hence, (1) holds for this case.

Case 4: Packet in state \( u \) reaches the sink after the newly arrived packet, and packet \( i+k \) in state \( z \) is either the newly arrived packet or reaches the sink after the newly arrived packet.

Suppose that the new packet is the \( n^{th} \) packet to leave the system according to state \( u \), and the \( n^{th} \) packet to leave the system according to state \( z \).

Since \( i > m \), we have \( a^u_i = a^z_{i-1} \). Similarly, when \( i+k > n \) and packet \( i+k \) in state \( z \) is either the newly arrived packet or will reach the sink after the newly arrived packet reaches (Figure 2).
we have \( t_{i+k}^z = t_{i+k-1}^z - 1 \).

We first show that \( t_{i+k}^z \geq t_{i+k-1}^z \) when \( i + k \geq n \).

For the base case, consider \( i + k = n \): Since \( t_{n}^z = t_{n-1}^z = t_{n-1} \), we have \( t_{n}^z \geq t_{n-1}^z \). Thus the result holds for \( i + k = n \).

Assume that the result holds for \( i + k = l > n \).

Consider \( i + k = l + 1 \): If \( t_{i+k}^z = \max(t_{i+1}^z + K + 1, d_{i+k}) \), then \( t_{i+l+1}^z \geq \max(t_{i+l}^z + K + 1, t_{i+l}^z) = t_{i+l}^z \), since \( d_{i+l+1} = d_{i+l} \) and \( t_{i+l}^z \geq t_{i+l}^z \). On the other hand, if \( t_{i+l}^z = t_{i+l}^z + d_{i+l+1} \), it follows that \( t_{i+l}^z = t_{i+l}^z + d_{i+l} \geq t_{i+l}^z + d_{i+l} = t_{i+l}^z \). Thus the result holds for \( l + 1 \).

Therefore, by induction, \( t_{i+k}^z \geq t_{i+k-1}^z \forall i + k \geq n \).

We now distinguish the cases where the newly arrived packet is located in the equivalent linear network.

Suppose that the newly arrived packet arrived in a node \( d \) in the equivalent linear network such that \( d \leq K \) (Figure 3).

We have the following cases.

- **Packet \( i \) in state \( u \) lies in a node between \( d \) and \( K + 1 \), and \( i + k \) in \( z \) also lies in a node between \( d \) and \( K + 1 \) in the equivalent linear network (Figure 3(a)): In this case, the arrival of the new packet increases the time for \( i \) and \( i + k \) to reach the sink by \( d \) slots. Therefore, \( t_{i+k}^u = t_{i+k}^u + d \) and \( t_{i+k}^z = t_{i+k-1}^z + d \). Since \( x < y \), (1) holds.

- **Packet \( i \) in state \( u \) lies in a node greater than \( K + 1 \), and \( i + k \) in \( z \) lies in a node between \( d \) and \( K + 1 \) in the equivalent linear network (Figure 3(b)): In this case, \( t_{i+k}^u \leq t_{i+k}^u + d \) and the situation is the same as in the previous case for packet \( i + k \). Hence, (1) holds.

- **Packet \( i \) in state \( u \) lies in a node between \( d \) and \( K + 1 \), and \( i + k \) in \( z \) lies in a node greater than \( K + 1 \) in the equivalent linear network (Figure 3(c)): In this case, we have \( t_{i+k}^u = t_{i+k}^u + d \) where \( d_{i+k}^u \leq K + 1 \) and \( d_{i+k}^z \geq t_{i+k}^z + K + 1 \). Hence, \( t_{i+k}^u > t_{i+k}^z \). Iteratively substituting \( i \) by \( i - 1 \), we either reach the newly arrived packet in state \( u \) or we reach a packet \( i + k \) in state \( z \) that lies in a node between \( d \) and \( K + 1 \) in the equivalent linear network. In the former case, by Case 3, we get a contradiction. For the latter case, we get a contradiction because of the previous case in this list. Hence, (1) holds.

- **Packet \( i \) in state \( u \) lies in a node greater than \( K + 1 \), and \( i + k \) in \( z \) also lies in a node greater than \( K + 1 \) in the equivalent linear network (Figure 3(d)): We can again prove (1) by contradiction. Suppose that \( t_{i+k}^u > t_{i+k}^z \) for some \( i \). We cannot have \( t_{i+k}^u = t_{i+k}^z \) since in that case \( d_{i+k}^u = d_{i+k}^z \), which contradicts \( x < y \). Hence, \( t_{i+k}^u = t_{i+k}^z + 1 \). Iteratively substituting \( i \) by \( i - 1 \), we either reach a situation where packet \( i \) is the newly arrived packet, or packet \( i + k \) is a packet that lies in a node \( b \leq K + 1 \). In either case, we obtain a contradiction since it falls under the previously listed scenarios. Hence, (1) holds.

Now, suppose that the newly arrived packet arrived in a node \( d \) in the equivalent linear network such that \( d \geq K \). We can again prove (1) by contradiction. Suppose that \( t_{i+k}^u > t_{i+k}^z \) for some \( i \). Since both \( i \) and \( i + k \) now lie in nodes \( d \geq K \), by a similar argument as in the last possibility above, we must have \( t_{i+k}^u = t_{i+k}^u + K + 1 > t_{i+k}^z + K + 1 \). Hence, \( t_{i+k}^u > t_{i+k}^z \). Again, by iteratively substituting \( i \) by \( i - 1 \), we either reach a situation where packet \( i \) is the newly arrived packet, or packet \( i + k \) is a packet that reaches the sink before the newly arrived packet. In the former case, we get a contradiction from Case 3, and in the latter case, we get a contradiction from Case 2. Thus, (1) holds in this case.

From these four cases, the result holds.