Supplementary Material

Proof of Lemma 4 for Class IV Trees

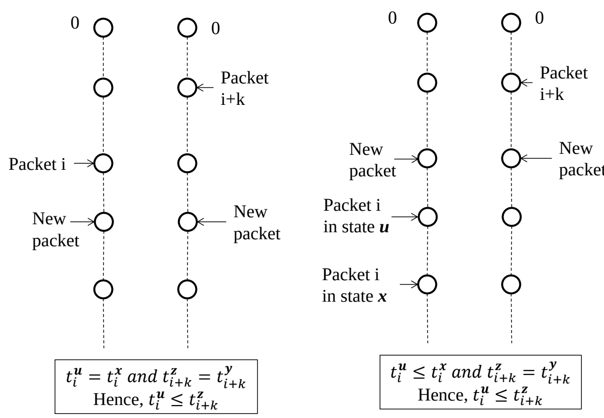
Lemma 4. *If we have $\boldsymbol{x} \prec \boldsymbol{y}$, then for all $j \in V$, we also have $\boldsymbol{x} + \boldsymbol{e}_j \prec \boldsymbol{y} + \boldsymbol{e}_j$.*

Proof: Let $\boldsymbol{u} = \boldsymbol{x} + \boldsymbol{e}_j$, and $\boldsymbol{z} = \boldsymbol{y} + \boldsymbol{e}_j$. Since $\boldsymbol{x} \prec \boldsymbol{y}$, $s(\boldsymbol{y}) - s(\boldsymbol{x}) = k \geq 0$. Thus, $s(\boldsymbol{z}) - s(\boldsymbol{u}) = k$.

We need to show that $\forall i = 1, 2, ..., s(\boldsymbol{x}) + 1$

$$t_i^{\boldsymbol{u}} \leq t_{i+k}^{\boldsymbol{z}} \tag{1}$$

We now consider the following cases. Recall that policy π_{IV} schedules the network with the new arrival as well.



(a) Lemma 4 Case 1 Example (b) Lemma 4 Case 2 Example

Fig. 1. Lemma 4 Examples

Case 1: Packet i in state $\boldsymbol{u}$ will reach the sink before the newly arrived packet, and packet $i+k$ in state $\boldsymbol{z}$ will also reach the sink before the newly arrived packet. Further, neither i nor $i+k$ is the newly arrived packet (see Figure 1(a)).

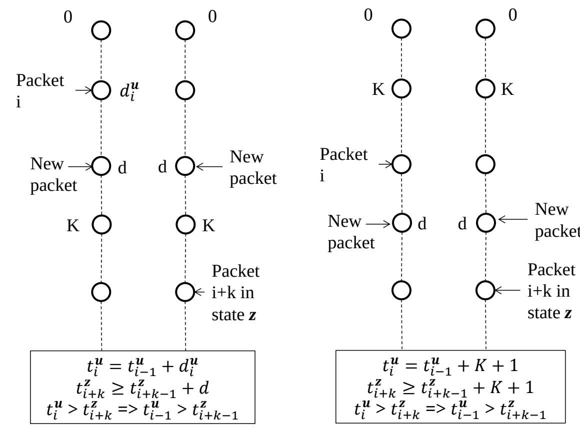
In this case, $t_i^{\boldsymbol{u}} = t_i^{\boldsymbol{x}}$ and $t_i^{\boldsymbol{z}} = t_i^{\boldsymbol{y}}$. Therefore, for all such packets, (1) holds.

Case 2: Packet i in state $\boldsymbol{u}$ is either the newly arrived packet or will reach the sink only after the newly arrived packet reaches, and packet $i+k$ in state $\boldsymbol{z}$ will reach the sink before the newly arrived packet and is not the newly arrived packet (see Figure 1(b)).

In this case, the i^{th} packet in state $\boldsymbol{x}$ becomes the $(i+1)^{th}$ packet in state $\boldsymbol{u}$. We show that $t_i^{\boldsymbol{u}} \leq t_i^{\boldsymbol{x}}$ as follows. It is easy to see that $d_i^{\boldsymbol{u}} \leq d_i^{\boldsymbol{x}}$. Therefore, by Lemma 2, $t_i^{\boldsymbol{u}} \leq t_i^{\boldsymbol{x}}$ if and only if $t_{i-1}^{\boldsymbol{u}} \leq t_{i-1}^{\boldsymbol{x}}$. Iteratively substitute i by $i-1$ until packet i is a packet that will reach the sink before the newly arrived packet. For this packet, from Case 1, we know that $t_i^{\boldsymbol{u}} = t_i^{\boldsymbol{x}}$. Hence, it follows that $t_i^{\boldsymbol{u}} \leq t_i^{\boldsymbol{x}}$ for all packets i that satisfy the condition in Case 2.

For packets in state $\boldsymbol{z}$, the situation is the same as in Case 1. Therefore, it follows that $t_i^{\boldsymbol{u}} \leq t_i^{\boldsymbol{x}} \leq t_{i+k}^{\boldsymbol{y}} = t_{i+k}^{\boldsymbol{z}}$.

Case 3: Packet i in state $\boldsymbol{u}$ is either the newly arrived packet or will reach the sink before the newly arrived packet reaches, and packet $i+k$ in state $\boldsymbol{z}$ is either the newly arrived packet or will reach the sink after the newly arrived packet reaches (Figure 2).



(a) New packet arrives at node $d \leq K$ (b) New packet arrives at node $d > K$

Fig. 2. Lemma 4 Case 3 Examples

We prove (1) by contradiction. Suppose that for some i, $t_i^{\boldsymbol{u}} > t_{i+k}^{\boldsymbol{z}}$.

Suppose that the newly arrived packet is in one of the first K nodes in the equivalent linear network, say, at node $d \leq K$ (Figure 2(a)). Then $t_i^{\boldsymbol{u}} = t_{i-1}^{\boldsymbol{u}} + d_i^{\boldsymbol{u}}$, and $t_{i+k}^{\boldsymbol{z}} \geq t_{i+k-1}^{\boldsymbol{z}} + d$. Hence, $t_i^{\boldsymbol{u}} > t_{i+k}^{\boldsymbol{z}}$ implies that $t_{i-1}^{\boldsymbol{u}} > t_{i+k-1}^{\boldsymbol{z}} + d - d_i^{\boldsymbol{u}} \geq t_{i+k-1}^{\boldsymbol{z}}$. By iteratively substituting i by $i-1$, we either obtain $i = 1$ in state $\boldsymbol{u}$ or $i+k$ is a packet that reaches the sink before the newly arrived packet according to state $\boldsymbol{z}$. If $i = 1$, $t_1^{\boldsymbol{u}} = d_1^{\boldsymbol{u}} \leq d \leq t_{k+1}^{\boldsymbol{z}}$, and thus we get a contradiction. If $i+k$ is a packet that reaches the sink before the newly arrived packet, then from Case 1, we get a contradiction. Hence, (1) must hold.

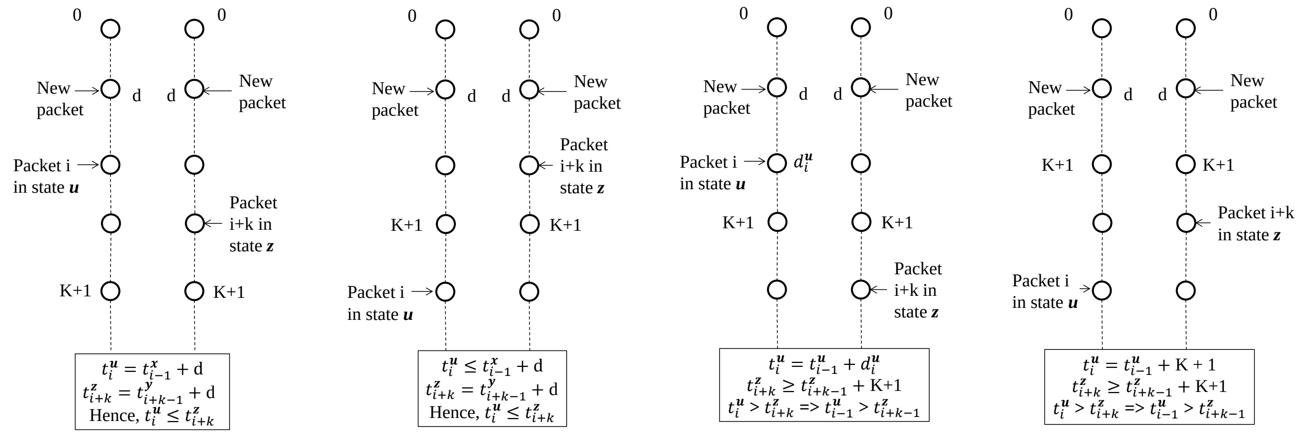
Suppose that the newly arrived packet arrives at a node $> K$ in the equivalent linear network, say, at node $d > K$ (Figure 2(b)). Suppose that for some i, $t_i^{\boldsymbol{u}} > t_{i+k}^{\boldsymbol{z}}$. If i is a packet that lies in one of the first K nodes in state $\boldsymbol{u}$, using the same argument as above, we get a contradiction. Otherwise, if $t_i^{\boldsymbol{u}} = d_i^{\boldsymbol{u}} \leq d$, since $t_{i+k}^{\boldsymbol{z}} \geq d$, we get a contradiction. If $t_i^{\boldsymbol{u}} = t_{i-1}^{\boldsymbol{u}} + K + 1 > t_{i+k}^{\boldsymbol{z}} \geq t_{i+k-1}^{\boldsymbol{z}} + K + 1$, this implies that $t_{i-1}^{\boldsymbol{u}} > t_{i+k-1}^{\boldsymbol{z}}$. Hence, iteratively substituting i by $i-1$, and arguing as above, we again get a contradiction.

Hence, (1) holds for this case.

Case 4: Packet i in state $\boldsymbol{u}$ reaches the sink after the newly arrived packet, and packet $i+k$ in state $\boldsymbol{z}$ is either the newly arrived packet or reaches the sink after the newly arrived packet.

Suppose that the new packet is the m^{th} packet to leave the system according to state $\boldsymbol{u}$, and the n^{th} packet to leave the system according to state $\boldsymbol{z}$.

Since $i > m$, we have $d_i^{\boldsymbol{u}} = d_{i-1}^{\boldsymbol{x}}$. Similarly, when $i+k > n$



(a) Packets i and $i+k$ between nodes d and $K+1$

(b) Packet i after node $K+1$, packet $i+k$ between nodes d and $K+1$

(c) Packet i between nodes d and $K+1$, packet $i+k$ after node $K+1$

(d) Packet i and $i+k$ after node $K+1$

Fig. 3. Lemma 4 Case 4 Examples

we have $d^{\boldsymbol{z}}_{i+k} = d^{\boldsymbol{y}}_{i+k-1}$.

We first show that $t^{\boldsymbol{z}}_{i+k} \geq t^{\boldsymbol{y}}_{i+k-1}$ when $i+k \geq n$.

For the base case, consider $i+k=n$: Since $t^{\boldsymbol{z}}_n \geq t^{\boldsymbol{z}}_{n-1} = t^{\boldsymbol{y}}_{n-1}$, we have $t^{\boldsymbol{z}}_n \geq t^{\boldsymbol{y}}_{n-1}$. Thus the result holds for $i+k=n$.

Assume that the result holds for $i+k=l>n$.

Consider $i+k=l+1$: If $t^{\boldsymbol{z}}_{l+1} = \max(t^{\boldsymbol{z}}_l + K+1, d^{\boldsymbol{z}}_{l+1})$, then $t^{\boldsymbol{z}}_{l+1} \geq \max(t^{\boldsymbol{y}}_{l-1}+K+1, d^{\boldsymbol{y}}_l) = t^{\boldsymbol{y}}_l$, since $d^{\boldsymbol{z}}_{l+1} = d^{\boldsymbol{y}}_l$ and $t^{\boldsymbol{z}}_l \geq t^{\boldsymbol{y}}_{l-1}$. On the other hand, if $t^{\boldsymbol{z}}_{l+1} = t^{\boldsymbol{z}}_l + d^{\boldsymbol{z}}_{l+1}$, it follows that $t^{\boldsymbol{z}}_{l+1} = t^{\boldsymbol{z}}_l + d^{\boldsymbol{z}}_{l+1} \geq t^{\boldsymbol{y}}_{l-1} + d^{\boldsymbol{y}}_l = t^{\boldsymbol{y}}_l$. Thus the result holds for $l+1$.

Therefore, by induction, $t^{\boldsymbol{z}}_{i+k} \geq t^{\boldsymbol{y}}_{i+k-1}\ \forall\ i+k \geq n$.

We now distinguish the cases where the newly arrived packet is located in the equivalent linear network.

Suppose that the newly arrived packet arrived in a node d in the equivalent linear network such that $d \leq K$ (Figure). We have the following cases.

- Packet i in state $\boldsymbol{u}$ lies in a node between d and $K+1$, and $i+k$ in $\boldsymbol{z}$ also lies in a node between d and $K+1$ in the equivalent linear network (Figure 3(a)): In this case, the arrival of the new packet increases the time for i and $i+k$ to reach the sink by d slots. Therefore, $t^{\boldsymbol{u}}_i = t^{\boldsymbol{x}}_{i-1}+d$ and $t^{\boldsymbol{z}}_{i+k} = t^{\boldsymbol{y}}_{i+k-1}+d$. Since $\boldsymbol{x} \prec \boldsymbol{y}$, (1) holds.
- Packet i in state $\boldsymbol{u}$ lies in a node $> K+1$, and $i+k$ in $\boldsymbol{z}$ lies in a node between d and $K+1$ in the equivalent linear network (Figure 3(b)): In this case, $t^{\boldsymbol{u}}_i \leq t^{\boldsymbol{x}}_{i-1}+d$ and the situation is the same as in the previous case for packet $i+k$. Hence, (1) holds.
- Packet i in state $\boldsymbol{u}$ lies in a node between d and $K+1$, and $i+k$ in $\boldsymbol{z}$ lies in a node $> K+1$ in the equivalent linear network (Figure 3(c)): In this case, we have $t^{\boldsymbol{u}}_i = t^{\boldsymbol{u}}_{i-1}+d^{\boldsymbol{u}}_i$ where $d^{\boldsymbol{u}}_i \leq K+1$, and $t^{\boldsymbol{z}}_{i+k} \geq t^{\boldsymbol{z}}_{i+k-1}+K+1$. Hence, if $t^{\boldsymbol{u}}_i > t^{\boldsymbol{z}}_{i+k}$, then $t^{\boldsymbol{u}}_{i-1} > t^{\boldsymbol{z}}_{i+k-1}$. Iteratively substituting i by $i-1$, we either reach the newly arrived packet in state $\boldsymbol{u}$ or we reach a packet $i+k$ in state $\boldsymbol{z}$ that lies in a node between d and $K+1$ in the equivalent linear network. In the former case, by Case 3, we get a contradiction. For the latter case, we get a contradiction because of the previous case in this list. Hence, (1) holds.
- Packet i in state $\boldsymbol{u}$ lies in a node $> K+1$, and $i+k$ in $\boldsymbol{z}$ also lies in a node $> K+1$ in the equivalent linear network (Figure 3(d)): We can again prove (1) by contradiction. Suppose that $t^{\boldsymbol{u}}_i > t^{\boldsymbol{z}}_{i+k}$ for some i. We cannot have $t^{\boldsymbol{u}}_i = d^{\boldsymbol{u}}_i$ since in that case $t^{\boldsymbol{x}}_{i-1} \geq d^{\boldsymbol{x}}_{i-1} = d^{\boldsymbol{u}}_i = t^{\boldsymbol{u}}_i > t^{\boldsymbol{z}}_{i+k} \geq t^{\boldsymbol{y}}_{i+k-1}$, which contradicts $\boldsymbol{x} \prec \boldsymbol{y}$. Hence we have $t^{\boldsymbol{u}}_i = t^{\boldsymbol{u}}_{i-1}+K+1 > t^{\boldsymbol{z}}_{i+k} \geq t^{\boldsymbol{z}}_{i+k-1}+K+1$. Hence, $t^{\boldsymbol{u}}_{i-1} > t^{\boldsymbol{z}}_{i+k-1}$. Iteratively substituting i by $i-1$, we either reach a situation where packet i is the newly arrived packet, or packet $i+k$ is a packet that lies in a node $b \leq K+1$. In either case, we obtain a contradiction since it falls under the previously listed scenarios. Hence, (1) holds.

Now, suppose that the newly arrived packet arrived in a node d in the equivalent linear network such that $d > K$. We can again prove (1) by contradiction. Suppose that $t^{\boldsymbol{u}}_i > t^{\boldsymbol{z}}_{i+k}$ for some i. Since both i and $i+k$ now lie in nodes $> K$, by a similar argument as in the last possibility above, we must have $t^{\boldsymbol{u}}_i = t^{\boldsymbol{u}}_{i-1}+K+1 > t^{\boldsymbol{z}}_{i+k} \geq t^{\boldsymbol{z}}_{i+k-1}+K+1$. Hence, $t^{\boldsymbol{u}}_{i-1} > t^{\boldsymbol{z}}_{i+k-1}$. Again, by iteratively substituting i by $i-1$, we either reach a situation where packet i is the newly arrived packet, or packet $i+k$ is a packet that reaches the sink before the newly arrived packet. In the former case, we get a contradiction from Case 3, and in the latter case, we get a contradiction from Case 2. Thus, (1) holds in this case.

From these four cases, the result holds. ■