Throughput of Rateless Codes over Broadcast Erasure Channels

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ABSTRACT

In this paper, we characterize the throughput of a broadcast network with n receivers using rateless codes with block size K. We assume that the underlying channel is a Markov modulated erasure channel that is *i.i.d.* across users, but can be correlated in time. We characterize the system throughput asymptotically in n. Specifically, we explicitly show how the throughput behaves for different values of the coding block size K as a function of n, as n approaches infinity. Under the more restrictive assumption of memoryless channels, we are able to provide a lower bound on the maximum achievable throughput for any finite values of K and n. Using simulations we show the tightness of the bound with respect to system parameters n and K, and find that its performance is significantly better than the previously known lower bound.

Categories and Subject Descriptors

C.2 [**Computer-Communication Networks**]: Miscellaneous

General Terms

Performance

Keywords

Broadcast erasure channel, Markov modulated channel, rateless erasure code, random linear network code, throughput, achievable rate

1. INTRODUCTION

In this work, we study the throughput of a wireless broadcast network with n receivers using rateless codes. In this broadcast network, channels between the transmitter and the receivers are modeled as packet erasure channels where transmitted packets may either be erased or be successfully received. This model describes a situation where packets may get lost or are not decodable at the receiver due to a variety of factors such as channel fading, interference or checksum errors. We assume that the underlying channel is a Markov modulated packet erasure channel that is *i.i.d.* across users, but can be correlated in time. We let γ denote the steady state probability that a packet is transmitted successfully on the erasure channel.

Instead of transmitting the broadcast data packet one after another through feedback and retransmissions, we investigate a class of coding schemes called rateless codes (or fountain codes). In this coding scheme, K broadcast packets are encoded together prior to transmission. K is called the coding block size. A rateless encoder views these K packets as K input symbols and can generate an arbitrary number of output symbols (which we call coded packets) as needed until the coding block is decoded. Although some coded packets may get lost during the transmission, rateless decoder can guarantee that any $K(1+\varepsilon)$ coded packets can recover the original K packets with high probability, where ε is a positive number that can be made arbitrarily small at the cost of coding complexity. Examples of rateless erasure codes include Raptor code [8], LT Code [4] and random linear network code [3], where the former two are used when K is very large and random linear network code is used when K is relatively small and the symbol space of packets is large. The best encoding and decoding complexity of rateless codes (e.g. Raptor codes) increase linearly as the coding block size K increases. Further, increasing the coding block size can result in large delays and large receiver buffer size. Therefore, real systems always have an upper bound on the value of K.

We consider broadcast traffic and a discrete time queueing model, where the numbers of packet arrivals over different time slots are independent and identically distributed and the packet length is a fixed value. We let λ denote the packet arrival rate and assume that the encoder waits until there are at least K packets in the queue and then encodes the first K of them as a single coding block. In this case, the largest arrival rate that can be stabilized is equal to the average number of packets that can be transmitted per slot, which we call the throughput. Therefore, we only need to characterize the throughput that can be achieved using rateless codes under parameters K and n. As described in Figure 1, the channel dynamics for the i^{th} receiver is denoted by a stochastic process $\{X_{ij}\}_{j\in\mathbb{N}}$, where j is the index of the time slot in which one packet can be transmitted and X_{ij} is the channel state of i^{th} receiver during the transmission of the j^{th} packet. We capture a fairly general correlation structure by letting the current channel state be impacted by the channel states in previous l time slots, where l can be any number. As the number of receivers n approaches infinity, we show that the throughput

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is nonzero only if the coding block size K increases at least as fast as $\log n$. In other words, if $c \triangleq \lim_{n\to\infty} \frac{K}{\log n}$, the asymptotic[1] throughput is positive whenever $c > 0$. In Theorem 1, by utilizing large deviation techniques, we give an explicit expression of the asymptotic throughput, which is a function of K, n, γ and the channel correlation structure.

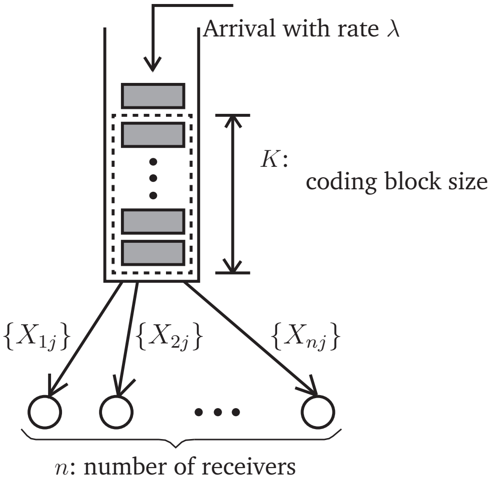

Figure 1: Broadcast with discrete time queueing model

To study the non-asymptotic behavior of the system, we make a more restrictive assumption that the channels are memoryless, which is a special case of the correlated channel model with $l = 0$. In other words, the erasure probability of every receiver channel at any time slot is $1-\gamma$. In this case, we show that, when $\frac{K}{\log n}$ is kept to be a constant, the throughput will follow a decreasing pattern as the number of receivers n increases. By combining this result with the characterization of the asymptotic throughput, we are able to provide a lower bound on the maximum achievable throughput for any finite values of K and n. This lower bound captures the asymptotic throughput in the sense that when n approaches infinity, it coincides with the asymptotic throughput.

1.1 Related Work

Among the works that investigate the throughput over erasure channels, [1], [9] and [7] are the most relevant to this work. In [9], the authors investigate the asymptotic throughput as a function of n and K and also show that the asymptotic throughput will be non-zero only if K at least scales with $\log n$. However, they only consider the channel correlation model with $l = 1$ and use a completely different proof technique. Moreover, no explicit expression on the asymptotic throughput is provided. In [1], a lower bound is provided on the maximum achievable rate λ of this paper. However, their bound does not converge to the asymptotic throughput when n approaches infinity. Further, their bound is only valid for $K > 16$ while our result is applicable for any values of K. Moreover, our bound is shown to be better in a variety of simulation settings with finite K and n, as will be showed in Section 5. In [7], the authors consider the case when instant feedback is provided from every user after the transmission of each decoded packets while we only assume that feedback is provided after the entire coding block has been decoded.

1.2 Key Contributions

The main contributions of this work are summarized as follows:

- We give an explicit expression of the asymptotic throughput of the system when the number of receivers n approaches infinity with different scales of K and n under the erasure channel with any levels of correlation. (Theorem 1)

- Under a more restrictive assumption that channels are memoryless, we reveal that when K grows with n in a way that the ratio $\frac{K}{\log n}$ is kept a constant, the throughput will follow a decreasing pattern as n increases, which tells us that for a quadratic increase of network size n, we need to have the coding block size K more than doubled in order to get the same throughput. (Theorem 2)

- We provide a lower bound on the maximum achievable throughput for any finite values of K and n under the memoryless channel assumption and show that its performance is significantly better than the previously known bound in [1]. (Theorem 3)

The rest of this paper is organized as follows. In Section 2 we describe our model and assumptions. In Section 3 we give the characterization of asymptotic throughput. In Section 4 we provide a lower bound on the maximum achievable rate for any finite values of K and n. In Section 5 we use simulations to verify our theoretical results. Detailed proofs on all the theorems can be found in Section 6. Finally, in Section 7, we conclude the paper.

2. SYSTEM MODEL

We consider a broadcast channel with n receivers. Time is slotted, and the numbers of broadcast packet arrivals over different time slots are i.i.d. with finite variance. We denote the expected number of packet arrivals per slot as the packet arrival rate λ. The transmission starts when there are more than K packets waiting in the incoming queue intended for all the receivers. Instead of transmitting these packets one after another using feedback and retransmissions, we view each data packet as a symbol and encode the first K of them into an arbitrary number of coded symbols as needed using rateless code (For example, Raptor Code [8] or random linear network code [3]) until the coding block is decoded. These K packets together form a single coding block with K being called block size. During the transmission, the coded symbols are transmitted one after another.

Each receiver send an ACK feedback signal after it has successfully decoded the K packets. In the following context, the term *packet* and *symbol* are used interchangeably.

We model the broadcast channel as a slotted broadcast packet erasure channel where one packet can be transmitted per slot. The channel dynamics can be represented by a stochastic process $\{X_{ij}\}_{1\le i\le n, j\in\mathbb{N}}$, where X_{ij} is the state of channel between transmitter and the i^{th} receiver during the transmission of j^{th} packet (we also call it the j^{th} time slot in the i^{th}

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[1] the asymptotic is with respect to increasing the number of receivers n

channel), which is given by

$$X_{ij} = \begin{cases} 1 & j^{\text{th}} \text{ packet in the } i^{\text{th}} \text{ channel is} \\ & \text{successfully received} \\ 0 & \text{otherwise} \end{cases}.$$

We assume that the dynamics of the channels for different receivers are independent and identical. More precisely, for all $1 \le i \le n$, $\{X_{ij}\}_{j \ge 1}$ are independent and identical processes.

Since, in practice, the channel dynamics are often temporarily correlated, we investigate the situation where the current channel state distribution depends on the channel states in the preceding l time slots. More specifically, for $\mathcal{F}_{im} = \{X_{ij}\}_{j \le m}$ and fixed l, we define $\mathcal{H}_{im} = \{X_{im}, \ldots, X_{i(m-l+1)}\}$ for $m \ge l \ge 1$ with $\mathcal{H}_{im} = \{\varnothing, \Omega\}$ for $l = 0$, and assume that $\mathbb{P}[X_{i(m+1)} = 1|\mathcal{F}_{im}] = \mathbb{P}[X_{i(m+1)} = 1|\mathcal{H}_{im}]$ for all $m \ge l$. To put it another way, when $l \ge 1$, the state $\big(X_{im}, \ldots, X_{i(m-l+1)}\big)$, $m \ge l$ forms a Markov chain. Denote by Π the transition matrix of the Markov chain $\big\{\big(X_{im}, \ldots, X_{i(m-l+1)}\big)\big\}_{m \ge l}$, where

$$\Pi = [\pi(s,u)]_{s,u \in \{0,1\}^l},$$

with $\pi(s,u)$ being the one-step transition probability from state s to state u. Throughout this paper, we assume that Π is irreducible and aperiodic, which ensures that this Markov chain is ergodic [6]. Therefore, for any initial value $\mathcal{H}_l$, the parameter γ_i is well defined and given by

$$\gamma_i = \lim_{m \to \infty} \mathbb{P}[X_{im} = 1],$$

and, from the ergodic theorem [6] we know

$$\mathbb{P}\left[\lim_{m \to \infty} \frac{\sum_{j=1}^{m} X_{ij}}{m} = \gamma_i\right] = 1.$$

Since $\{X_{ij}\}_{j \ge 1}$ for all $1 \le i \le n$ are i.i.d., we denote $\gamma = \gamma_i$, for all $1 \le i \le n$.

Using near optimal rateless codes, such as Raptor Code [8], LT Code [4] and random linear network code [3], only slightly more than K coded symbols are needed to decode the whole coding block. For simplicity, here we assume that any combination of K coded symbols can lead to a successful decoding of the K packets.

According to the above system model, we have the following definitions:

DEFINITION 1. *The number of time slots (number of transmitted coded symbols) needed for user i to successfully decode K packets is defined as*

$$T_i(K) = \min_m \left\{ \sum_{j=1}^{m} X_{ij} \ge K \right\}.$$

DEFINITION 2. *The number of time slots (number of transmitted coded symbols) needed to complete the transmission of a single coding block to all the receivers is defined as*

$$T(n,K) = \max\{T_i(K), i = 1, 2, \ldots, n\}.$$

DEFINITION 3 (INITIAL STATE). *Since the current channel state depends on the channel states in the previous l time-slots, for each receiver i, by assuming that the system starts at time slot 1, we define the initial state of receiver i as*

$$\mathcal{E}_i = \left\{X_{i(-l+1)}, X_{i(-l+2)}, \ldots, X_{i0}\right\}.$$

The initial states for all the receivers is then denoted as $\mathcal{E} \triangleq \cup_{i=1}^{n} \mathcal{E}_i$.

DEFINITION 4 (THROUGHPUT). *Under an initial state $\mathcal{E}$, the average number of packets that can be successfully transmitted per slot is defined as*

$$\eta(n,K,\mathcal{E}) = \frac{K}{\mathbb{E}[T(n,K)|\mathcal{E}]},$$

which we call the throughput under initial condition $\mathcal{E}$. In the special case where the channel states are i.i.d. ($l = 0$), we know that $\mathcal{E} = \varnothing$ and we can denote the throughput as

$$\eta(n,K) = \frac{K}{\mathbb{E}[T(n,K)]}. \tag{1}$$

3. ASYMPTOTIC THROUGHPUT

Before presenting the main results, we need to introduce some necessary definitions. First, define a mapping f from the state space of the Markov chain $\{0,1\}^l$ to $\{0,1\}$ as

$$f\big((X_{im}, \ldots, X_{i(m-l+1)})\big) = X_{im}.$$

Then, given a real number θ, we define a matrix Π_θ as

$$\Pi_\theta = \begin{cases} \left[\pi(s,u)e^{\theta f(u)}\right]_{s,u \in \{0,1\}^l} & \text{when } l \ge 1 \\ \left[\gamma e^{\theta}\right] & \text{when } l = 0 \end{cases}.$$

Last, define a standard large deviation rate function $\Lambda(\beta, \Pi)$ as

$$\Lambda(\beta,\Pi) = \sup_\theta\{\theta\beta - \log\rho(\Pi_\theta)\}, \tag{2}$$

where $\rho(\Pi_\theta)$ denotes the Perron-Frobenious eigenvalue of Π_θ (See Theorem 3.1.1 in [2]), which is the largest eigenvalue of Π_θ.

The asymptotic throughput for any values of K as a function of n under any initial condition $\mathcal{E}$ is characterized by the theorem below:

THEOREM 1. *Assume that K is a function of n and the value of $\lim_{n\to\infty} \frac{K}{\log n}$ exists, which we denote as $c \triangleq \lim_{n\to\infty} \frac{K}{\log n}$, then for any initial state $\mathcal{E}$ we have*

$$\lim_{n\to\infty} \eta(n,K,\mathcal{E}) = \sup\left\{\beta \middle| c \ge \frac{\beta}{\Lambda(\beta,\Pi)}, 0 \le \beta < \gamma\right\}. \tag{3}$$

PROOF. see Section 6.1. ☐

From Theorem 1, we know that, if the coding block size K is set to be a function of the network size n, then we can characterize the asymptotic throughput when n approaches infinity in an explicit form. Equation (3) shows that the asymptotic throughput is irrelevant to the initial state $\mathcal{E}$ and is a function of γ, $\lim_{n\to\infty} \frac{K}{\log n}$ and the channel correlation structure indicated by Π.

By Theorem 1, the asymptotic throughput in the special cases when $K \in o(\log n)$ and $K \in \omega(\log n)$ are given in the following corollary.

COROLLARY 1.1. *Assume K is a function of n, for any initial state $\mathcal{E}$ we have*

1. *if $K \in o(\log n)$, then*[2]

$$\lim_{n\to\infty} \eta(n,K,\mathcal{E}) = 0.$$

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[2] We use standard notations: $f(n) = o(g(n))$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$ and $f(n) = \omega(g(n))$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ diverges

2. if $K \in \omega(\log n)$, then

$$\lim_{n\to\infty} \eta(n, K, \mathcal{E}) = \gamma.$$

PROOF. 1) If $K \in o(\log n)$, then $c = \lim_{n\to\infty} \frac{K}{\log n} = 0$ and we have

$$\left\{\beta \middle| c \geq \frac{\beta}{\Lambda(\beta,\Pi)}, 0 \leq \beta < \gamma\right\} = \{0\}.$$

According to Theorem 1, we get

$$\lim_{n\to\infty} \eta(n, K, \mathcal{E}) = \sup\{0\} = 0.$$

2) If $K \in \omega(\log n)$, then $c = \lim_{n\to\infty} \frac{K}{\log n} = \infty$ and we have

$$\left\{\beta \middle| c \geq \frac{\beta}{\Lambda(\beta,\Pi)}, 0 \leq \beta < \gamma\right\} = [0, \gamma).$$

According to Theorem 1, we get

$$\lim_{n\to\infty} \eta(n, K, \mathcal{E}) = \sup[0, \gamma) = \gamma.$$

□

Corollary 1.1 says that the throughput will vanish to 0 as the increase of n, when K does not scale as fast as $\log n$. Whereas when K scales faster than $\log n$ (Or more specifically, when $K \in \omega(\log n)$), the capacity of this system γ can be approached. It should be noted that Theorem 1, together with Corollary 1.1, are a generalized version of Theorem 1 in [9], which only consider the case when $l = 1$ and does not give the explicit expression of asymptotic throughput.

For the special case when the channels are memoryless ($l = 0$), we can compute $\Lambda(\beta, \Pi)$ explicitly, as shown in the corollary below

COROLLARY 1.2. *Assume that K is function of n and the channels are memoryless ($l = 0$), we have if $\lim_{n\to\infty} \frac{K}{\log n} = c$, where c is a positive constant, then*

$$\lim_{n\to\infty} \eta(n, K) = \sup\left\{\beta \middle| \log\frac{\beta}{\gamma} + \frac{1-\beta}{\beta}\log\frac{1-\beta}{1-\gamma} \geq \frac{1}{c}, 0 \leq \beta < \gamma\right\}. \quad (4)$$

PROOF. When $l = 0$, $\Pi_\theta = [\gamma e^\theta]$ is a degenerate matrix with a single entry and $\rho(\Pi_\theta) = \rho(\gamma e^\theta) = \gamma e^\theta$. Therefore we have, according to Equation (2)

$$\Lambda(\beta, \Pi) = \beta \log\frac{\beta}{\gamma} + (1-\beta)\log\frac{1-\beta}{1-\gamma}.$$

□

4. MAXIMUM ACHIEVABLE THROUGHPUT

For all rateless coding schemes, the encoding and decoding complexity increases linearly in K, the size of the coding block. Moreover, the value of K determines the receiver buffer size. Therefore, in reality, the value of K is often limited by the decoder buffer size or the computational power of both sender and receiver. Then we have to consider the case when K is finite and need to answer the following questions: For a given number of receivers n, channel statistics, and a maximum available coding block size K, what is maximum packet arrival rate λ that can be supported by this given system? For a specific number of receivers and channel statistics, if we are given a target packet arrival rate λ, how can we design the value of K in the system such that the target arrival rate can be supported?

In order to answer these questions, we make a more restrictive assumption in this section that channels are memoryless, meaning that the channel states are i.i.d. across different time slots. Based on this assumption, in the theorem below, we are able to show that when we keep the value of $K/\log n$ to be a constant as we increase K or n, the throughput will follow a decreasing pattern. We leave the case when $l > 0$ for future study.

THEOREM 2. *When the channels are memoryless ($l = 0$), for any $n \in \mathbb{N}$, $K \in \mathbb{N}$ and $\alpha \in \mathbb{N}$, we have*

$$\eta(n^\alpha, \alpha K) < \eta(n, K).$$

REMARK 2.1. *While Theorem 1 tells us that in order to achieve a nonzero throughput, we can double the coding block size K for every quadratic increase of n, which is to make $K/\log n$ a fixed value, it does not tell us anything about how the throughput will converge as n approaches infinity. This theorem indicates that under the memoryless channel assumption, if we adapt the coding block size K with the increase of network size n in a way that $K/\log n$ is kept as a fixed value, then the throughput will follow a decreasing pattern before it reaches the asymptotic throughput.*

REMARK 2.2. *For the case when the channels are correlated, the expected transmission time for a single coding block varies under different initial states $\mathcal{E}$. It is technically difficult to obtain the steady state distribution of initial state $\mathcal{E}$ across different transmissions. Therefore, it is not clear whether this inequality will hold for $l > 0$.*

PROOF. see Section 6.2. □

By the help of the above theorem, we can get a lower bound on the maximum stable throughput that can be achieved for any finite values of coding block size K and network size n, as shown in the theorem below.

THEOREM 3. *For a broadcast network with n receivers, coding block size K and packet arrive rate λ, when the erasure broadcast channels are memoryless ($l = 0$) with erasure probability $1 - \gamma$, the system is stable if*

$$\lambda \leq \mathcal{R}\left(\frac{\log n}{K}\right),$$

where

$$\mathcal{R}(r) = \sup\left\{\beta \middle| \log\frac{\beta}{\gamma} + \frac{1-\beta}{\beta}\log\frac{1-\beta}{1-\gamma} \geq r, 0 \leq \beta < \gamma\right\}.$$

PROOF. From Equation (10) in Lemma 1 we can see that when K and n are finite, the transmission time of a coding block $T(n, K)$ is light-tail distributed, meaning that it has finite variance. Then according to [5] we know that the queue will be stable if the traffic intensity of this queue, which is defined as the packet arrival rate λ over the service rate, is less than 1. Therefore, the queue will be stable if the arrival rate λ satisfies

$$\lambda < \sup\left\{\mu \middle| \frac{\mu}{K/\mathbb{E}[T(n,K)]} < 1\right\} = \frac{K}{\mathbb{E}[T(n,K)]} = \eta(n, K). \quad (5)$$

By Theorem 2 we know that $\eta(n, K) > \eta(n^\alpha, \alpha K)$ for any integer values of α, therefore we have

$$\eta(n, K) > \lim_{\alpha\to\infty} \eta(n^\alpha, \alpha K). \tag{6}$$

Since $\frac{\alpha K}{\log n^\alpha} = \frac{K}{\log n}$ for any value of α, then by applying Theorem 1 and Corollary 1.2, we get

$$\begin{aligned} &\lim_{\alpha\to\infty} \eta(n^\alpha, \alpha K) = \\ &\sup\left\{\beta \middle| \log\frac{\beta}{\gamma} + \frac{1-\beta}{\beta}\log\frac{1-\beta}{1-\gamma} \geq \frac{\log n}{K}, 0 \leq \beta < \gamma\right\} \\ &= \mathcal{R}\left(\frac{\log n}{K}\right), \end{aligned}$$

which, by combining Equation (5) and Equation (6), completes the proof. $\square$

In order to compare this lower bound on the maximum achievable rate with the existing bound given in [1], we restate Theorem 2 in [1] as the following

THEOREM 4 (THEOREM 2 IN [1]). *For a broadcast network with n receivers, coding block length $K > 16$ and packet arrive rate λ, when the erasure broadcast channels are memoryless ($l = 0$) with erasure probability $1-\gamma$, the system is stable if*

$$\lambda < \frac{(1-\gamma)K}{K + (\log n + 0.78)\sqrt{K} + 2.61}. \tag{7}$$

For the ease of notation let us denote the bounds given in Theorem 4 as *CSE bound* using the initials of the authors' last name.

Firstly we should notice that CSE bound is only valid when $K > 16$, while there is no such restriction for our bound. Secondly, our bound converges to the asymptotic throughput in the sense that as n approaches infinity while keeping $K/\log n$ as a constant c, our bound on the maximum achievable rate will converge to the asymptotic throughput with parameter c. Or more specifically,

$$\lim_{n\to\infty} \eta(n, K) = \lim_{n\to\infty} \mathcal{R}\left(\frac{\log n}{K}\right) = \mathcal{R}\left(\frac{1}{c}\right), \tag{8}$$

which can be seen from Theorem 1 and Theorem 3. However, in the CSE bound, when we keep the ratio $K/\log n$ to be a constant c, as n or K approaches infinity, the bound will become trial (approach 0), which can be seen from the equation below.

$$\begin{aligned} &\lim_{K\to\infty} \frac{(1-\gamma)K}{K + (\log n + 0.78)\sqrt{K} + 2.61} \\ =&\lim_{K\to\infty} \frac{(1-\gamma)}{1 + (1/c + 0.78/K)\sqrt{K} + 2.61/K} \\ =&0. \end{aligned} \tag{9}$$

Next, in Section 5, we show that our bound outperforms CSE bound under various simulation settings.

5. SIMULATION

In this Section, we conduct simulation experiments to verify our main results.

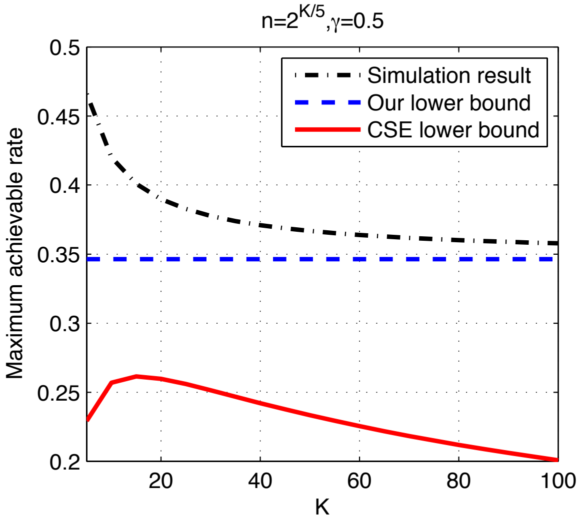


Figure 2: Illustration for example 1

5.1 Example 1

This example verifies both Theorem 1 and Theorem 2. We choose a memoryless channel with $\gamma = 0.5$. By keeping $K/\log n$ as a constant $5/\log 2$, we change K from 5 to 100 and calculate the maximum achievable rate, which is $\eta(n, K)$, through simulations for each pair of (K, n). Since the value of our bound is a function of the ratio $K/\log n$, in this case, it is a constant for all K and is equal to the asymptotic throughput with a parameter $5/\log 2$. From Figure 2 we can see that as K approaches infinity, the maximum achievable rate converges to our lower bound (which is also the asymptotic throughput in this case) in a decreasing manner, which validates both Theorem 1 and Theorem 2.

In this case, we also plotted the CSE lower bound given by Theorem 4 and we can see that CSE bound gradually approaches zero as indicated by Equation (9) while our bound is a constant value and is asymptotically tight.

5.2 Example 2

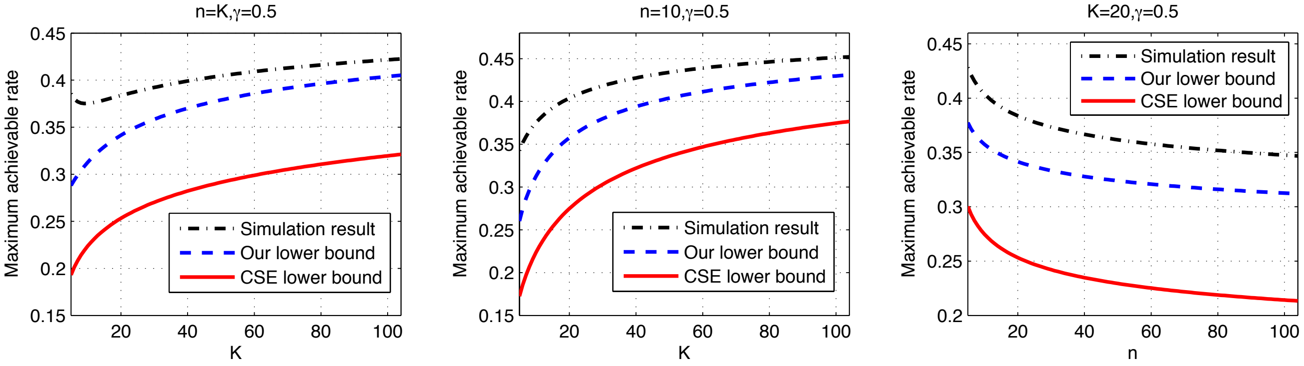
In this example, we conduct three set of experiments with different values of K as a function of n and show that our bound outperforms CSE bound in all these simulation settings.

In the first case, we set the coding block size K to be the same as the network size n and change n from 5 to 100. We plot the simulation result of the maximum achievable rate as well as our bound and CSE bound in Figure 3a, since in this case K scales faster than $\log n$, the achievable rate will approach system capacity γ as the network size n grows.

In the second case, we assume that the number of receivers is fixed to be 10 and we increase coding block size K from 5 to 100. The simulations result, together with the two bounds, are plotted in Figure 3b. In this case, the achievable rate will also approach system capacity γ as n increases.

In the final case, as shown in Figure 3c, we keep the coding block size to be a constant 20 and increase the number of receivers from 5 to 100. Since K does not increase with $\log n$ at all, the achievable rate will vanish to 0 as n grows.

From Figure 2, 3a, 3b and 3c, we can see that our bound



(a) n=K (b) n=10 (c) K=20

Figure 3: Illustration for example 2

obtained by Theorem 3 is significantly better than the lower bound achieved in [1] in all these four different cases.

6. PROOFS

6.1 Proof of Theorem 1

In order to prove Theorem 1, we first need the following lemmas (Lemma 1, Lemma 2 and Lemma 3).

LEMMA 1. *For any $\beta \in (0,1)$ and any values of $\mathcal{E}$, we have*

$$\begin{aligned}&\mathbb{P}\left[T(n,K) > \frac{k}{\beta}\middle| K=k,\mathcal{E}\right]\\ =&1-\left(1-e^{-\frac{k}{\beta}\Lambda(\beta,\Pi)\mathbf{1}(\beta<\gamma)+g(\beta,k,\mathcal{E})}\right)^n,\end{aligned} \qquad (10)$$

where

$$g(\beta,k,\mathcal{E}) \in \left\{\begin{array}{ll} o(k) \text{ as } k\to\infty & \text{if } \beta<\gamma \\ o(1) \text{ as } k\to\infty & \text{if } \beta>\gamma \end{array}\right. .$$

PROOF OF LEMMA (1). From definition (1) and (2), we have, for any t,

$$\{T(n,K)\le t,\mathcal{E}\} = \bigcap_{i=1}^{n}\{T_i(K)\le t,\mathcal{E}_i\}.$$

Therefore, we have

$$\begin{aligned}&\mathbb{P}[T(n,K)>t|K=k,\mathcal{E}]\\ =&1-\mathbb{P}[T(n,K)\le t|K=k,\mathcal{E}]\\ =&1-\prod_{i=1}^{n}\left(1-\mathbb{P}[T_i(K)>t|K=k,\mathcal{E}_i]\right).\end{aligned} \qquad (11)$$

Let $t=\frac{k}{\beta}$, from definition 1 we can get, for any $1\le i\le n$,

$$\begin{aligned}\mathbb{P}\left[T_i(K)>\frac{k}{\beta}\middle|K=k,\mathcal{E}_i\right] =&\mathbb{P}\left[\sum_{j=1}^{k/\beta}X_{ij}<k\middle|\mathcal{E}_i\right]\\ =&\mathbb{P}\left[\frac{\sum_{j=1}^{k/\beta}X_{ij}}{k/\beta}<\beta\middle|\mathcal{E}_i\right].\end{aligned}$$

and

$$\lim_{k\to\infty}\frac{\log\mathbb{P}\left[T_i(K)>\frac{k}{\beta}\middle|K=k,\mathcal{E}_i\right]}{k/\beta} = -\Lambda(\beta,\Pi)\mathbf{1}(\beta<\gamma), \qquad (12)$$

with the last equation being a direct application of Theorem 3.1.2 in [2] (Gärtner-Ellis Theorem for finite state Markov chains). Notice that the right hand side of Equation (12) is fixed for all possible values of i and $\mathcal{E}_i$ as long as the values of β and Π are fixed. Then the proof completes by combining (11) and (12). ☐

LEMMA 2. *Assume k is a function of n and denote $k:=k(n)$, and define $f(k,\beta,\mathcal{E}):=e^{\frac{k}{\beta}\Lambda(\beta,\Pi)\mathbf{1}(\beta<\gamma)-g(\beta,k,\mathcal{E})}$, then we have*

1. *For a fixed $\beta\in(0,1)$, if $\lim_{n\to\infty}\frac{n}{f(k(n),\beta,\mathcal{E})}=0$, then*

$$\lim_{n\to\infty}\mathbb{P}\left[T(n,K)>\frac{k(n)}{\beta}\middle|K=k(n),\mathcal{E}\right]=0. \qquad (13)$$

2. *For a fixed $\beta\in(0,1)$, if $\lim_{n\to\infty}\frac{n}{f(k(n),\beta,\mathcal{E})}=\infty$, then*

$$\lim_{n\to\infty}\mathbb{P}\left[T(n,K)>\frac{k(n)}{\beta}\middle|K=k(n),\mathcal{E}\right]=1. \qquad (14)$$

PROOF OF LEMMA 2. According to Lemma 1 and the definition of $f(k(n),\beta,\mathcal{E})$, we have

$$\begin{aligned}&\mathbb{P}\left[T(n,K)>\frac{k(n)}{\beta}\middle|K=k(n),\mathcal{E}\right]\\ =&1-\left(1-\frac{1}{f(k(n),\beta,\mathcal{E})}\right)^n\\ =&1-\left[\left(1-\frac{1}{f(k(n),\beta,\mathcal{E})}\right)^{f(k(n),\beta,\mathcal{E})}\right]^{\frac{n}{f(k(n),\beta,\mathcal{E})}}.\end{aligned}$$

Since the function $\left(1-\frac{1}{x}\right)^x$ with domain $(1,+\infty)$ is a bounded and strictly increasing function with region $(0,e^{-1})$ and the fact that $f(k,\beta)>1$, we know that if $\lim_{n\to\infty}\frac{n}{f(k(n),\beta,\mathcal{E})}=$

∞, then

$$\begin{aligned}&\liminf_{n\to\infty}\mathbb{P}\left[T(n,K)>\frac{k(n)}{\beta}\Big|K=k(n),\mathcal{E}\right]\\=&1-\limsup_{n\to\infty}\left[\left(1-\frac{1}{f(k(n),\beta,\mathcal{E})}\right)^{f(k(n),\beta,\mathcal{E})}\right]^{\frac{n}{f(k(n),\beta,\mathcal{E})}}\\\geq&1-\limsup_{n\to\infty}e^{-\frac{n}{f(k(n),\beta,\mathcal{E})}}\\=&1,\end{aligned}$$

which, together with the fact that $\mathbb{P}\big[T(n,K)>\frac{k(n)}{\beta}\big|K=k(n),\mathcal{E}\big]\leq 1$, yields Equation (14).

If $\lim_{n\to\infty}\frac{n}{f(k(n),\beta,\mathcal{E})}=0$, then $f(k(n),\beta,\mathcal{E})\to\infty$ as $n\to\infty$, which results in

$$\lim_{n\to\infty}\left(1-\frac{1}{f(k(n),\beta,\mathcal{E})}\right)^{f(k(n),\beta,\mathcal{E})}=e^{-1}.$$

Then we can obtain

$$\begin{aligned}&\limsup_{n\to\infty}\mathbb{P}\left[T(n,K)>\frac{k(n)}{\beta}\Big|K=k(n),\mathcal{E}\right]\\=&1-\liminf_{n\to\infty}\left[\left(1-\frac{1}{f(k(n),\beta,\mathcal{E})}\right)^{f(k(n),\beta,\mathcal{E})}\right]^{\frac{n}{f(k(n),\beta,\mathcal{E})}}\\=&1-\\&\liminf_{n\to\infty}\left[\lim_{n\to\infty}\left(1-\frac{1}{f(k(n),\beta,\mathcal{E})}\right)^{f(k(n),\beta,\mathcal{E})}\right]^{\frac{n}{f(k(n),\beta,\mathcal{E})}}\\=&1-1=0,\end{aligned}$$

which leads to Equation (13). ☐

Lemma 3. *Let $\{h_n(x)\}$ be a set of Lebesgue measurable functions defined on $[0,\infty)$ and $h_n(x)$ converges to $\mathbf{1}(x<y)$ almost everywhere for some $y>0$. If $h_n(x)$ is a decreasing function of x and have the range $[0,1]$ for any $n\in\mathbb{N}$, then $h_n(x)$ converges globally in measure to $\mathbf{1}(x<y)$.*

Proof. Choose $\varepsilon>0$. Since $h_n(x)$ converges to $\mathbf{1}(x<y)$ almost everywhere, for any $\delta>0$, we can find $N\in\mathbb{N}$ such that for any $n>N$, we have

$$\begin{aligned}|h_n\left(y-\delta/2\right)-1|<\varepsilon\\|h_n\left(y+\delta/2\right)-0|<\varepsilon.\end{aligned}$$

Since $0\leq h_n(x)\leq 1$ for any $x\in[0,\infty)$ and $h_n(x)$ is a decreasing function of x, we know that, for any $n>M$,

$$\begin{aligned}h_n(x)&>1-\varepsilon&&\forall x<y-\delta/2\\h_n(x)&<\varepsilon&&\forall x>y+\delta/2.\end{aligned}$$

Therefore, for any $n>N$,

$$\begin{aligned}&\nu\left(\{|h_n(x)-\mathbf{1}(x<y)|>\varepsilon\}\right)\\<&\nu([y-\delta/2,y])+\nu([y,y+\delta/2])=\delta,\end{aligned}$$

where ν is the Lebesgue measure. Since ε and δ are arbitrarily chosen, from the above inequality we know that $h_n(x)$ converges globally in measure to $\mathbf{1}(x<y)$. ☐

With Lemma 1,2 and 3 established, we now turn to the proof of Theorem 1.

Proof of Theorem 1. Since K is assumed to be a function of n, we denote this function as $k(n)$. According to definition 4 we have, for any values of $\mathcal{E}$,

$$\begin{aligned}&\lim_{n\to\infty}\left(\eta(n,K,\mathcal{E})\right)^{-1}\\=&\lim_{n\to\infty}\mathbb{E}\left[\frac{T(n,K)}{K}\Big|\mathcal{E}\right]\\=&\lim_{n\to\infty}\int_0^\infty\frac{\mathbb{P}\left[T(n,K)>s|K=k(n),\mathcal{E}\right]}{k(n)}ds\\=&\lim_{n\to\infty}\int_0^\infty\mathbb{P}\left[T(n,K)>k(n)u|K=k(n),\mathcal{E}\right]du.\end{aligned}\tag{15}$$

According to the assumption that $\lim_{n\to\infty}k(n)/\log(n)=c$, we have

$$\begin{aligned}&\lim_{n\to\infty}\frac{n}{e^{\frac{k(n)}{\beta}\Lambda(\beta,\Pi)\mathbf{1}(\beta<\gamma)-g(\beta,k,\mathcal{E})}}\\=&\begin{cases}0&c>\frac{\beta}{\Lambda(\beta,\Pi)\mathbf{1}(\beta<\gamma)}\\\infty&c<\frac{\beta}{\Lambda(\beta,\Pi)\mathbf{1}(\beta<\gamma)}\end{cases}.\end{aligned}$$

Since $\frac{\beta}{\Lambda(\beta,\Pi)\mathbf{1}(\beta<\gamma)}|_{\beta=0}=0$, $\lim_{\beta\to\gamma^-}\frac{\beta}{\Lambda(\beta,\Pi)\mathbf{1}(\beta<\gamma)}=\infty$ and $\frac{\beta}{\Lambda(\beta,\Pi)\mathbf{1}(\beta<\gamma)}$ is a monotone increasing function on the domain $(0,\gamma)$, the equation $c=\frac{\beta}{\Lambda(\beta,\Pi)\mathbf{1}(\beta<\gamma)}$ has only one solution of β, which we denote as

$$\beta_c=\sup\left\{\beta\Big|c\geq\frac{\beta}{\Lambda(\beta,\Pi)},0\leq\beta<\gamma\right\}.$$

Then by Lemma 2 we get

$$\begin{aligned}&\lim_{n\to\infty}\mathbb{P}\left[T(n,K)>k(n)u\big|K=k(n),\mathcal{E}\right]\\=&\begin{cases}1&\text{if }u<\frac{1}{\beta_c}\\0&\text{if }u>\frac{1}{\beta_c}\end{cases}.\end{aligned}\tag{16}$$

Let us denote $h_n(u)=\mathbb{P}\big[T(n,K)>k(n)u\big|K=k(n),\mathcal{E}\big]$. Equation (16) implies that $h_n(u)$ converges to $\mathbf{1}(u<1/\beta_c)$ pointwisely. Since $h_n(u)$ is a decreasing function of u and has the range $[0,1]$ for all n, by Lemma 3 we know that $h_n(u)$ globally converges in measure to $\mathbf{1}(u<1/\beta_c)$. We also know that the set of function $\{h_n(u)\}$ is uniformly bounded. Then we can apply Vitali convergence theory to Equation (15) to exchange the limit and integral and obtain

$$\begin{aligned}&\lim_{n\to\infty}\mathbb{E}\left[\frac{T(n,K)}{K}\Big|\mathcal{E}\right]\\=&\int_0^\infty\lim_{n\to\infty}\mathbb{P}\left[T(n,K)>k(n)u\big|K=k(n),\mathcal{E}\right]du=\frac{1}{\beta_c},\end{aligned}$$

which, by combining Equation (15), completes the proof. ☐

6.2 Proof of Theorem 2.

Proof. When the channel states are i.i.d., according to definition 1 and 2, $T(n,K)$ and $T(n^\alpha,\alpha K)$ can be expressed as

$$T(n,K)=\max_{1\leq i\leq n}\sum_{j=1}^{K}T_{ij}$$

$$T(n^\alpha,\alpha K)=\max_{1\leq i\leq n^\alpha}\sum_{j=1}^{\alpha K}T_{ij},$$

where $\{T_{ij}\}_{i\in\mathbb{N},j\in\mathbb{N}}$ are i.i.d. geometric random variables with parameter γ. Let us deonte

$$S_i^r = \sum_{j=1+(r-1)K}^{rK} T_{ij}.$$

Then we know that $\{S_i^r\}_{i\in\mathbb{N},r\in\mathbb{N}}$ are also i.i.d. random variables. The above two equations can be rewritten as

$$T(n,K) = \max_{1\leq i\leq n} S_i^1 \tag{17}$$

$$T(n^\alpha,\alpha K) = \max_{1\leq i\leq n^\alpha} \sum_{r=1}^{\alpha} S_i^r. \tag{18}$$

Instead of viewing Equation (18) as a 1-dimensional maximization over n^α points, we can think of it as an α-dimensional maximization over n^α points where we can choose coordinate from 1 to n on each dimension and therefore can further rewrite Equation (18) as

$$T(n^\alpha,\alpha K) = \max_{1\leq i_1\leq n} \max_{1\leq i_2\leq n} \cdots \max_{1\leq i_\alpha\leq n} \sum_{r=1}^{\alpha} S^r_{(i_1,i_2,\ldots,i_\alpha)}, \tag{19}$$

where

$$S^r_{(i_1,i_2,\ldots,i_\alpha)} = S^r_{\sum_{u=1}^{\alpha} n^{u-1}(i_u-1)+1}$$

and i_u can be viewed as the coordinate in the u^{th} dimension.

Next, we are going to use Equation (19) to build a lower bound for the expection of $T(n^\alpha,\alpha K)$.

For fixed values of $i_2,i_3,\ldots,i_\alpha$, let us find a i_1^* such that

$$i_1^*\left(i_2,\ldots,i_\alpha\right) = \arg\max_{1\leq i_1\leq n} S^1_{(i_1,i_2,\ldots,i_\alpha)}, \tag{20}$$

which we denote as i_1^* for short. Then according to Equation (19), we can find a lower bound for $\mathbb{E}[T(n^\alpha,\alpha K)]$ by choosing $i_1 = i_1^*\left(i_2,\ldots,i_\alpha\right)$ for all possible values of $i_2,i_3,\ldots,i_\alpha$, which is

$$\begin{aligned}
&\mathbb{E}\left[T(n^\alpha,\alpha K)\right]\\
=&\mathbb{E}\left[\max_{1\leq i_1\leq n}\max_{1\leq i_2\leq n}\cdots\max_{1\leq i_\alpha\leq n}\sum_{r=1}^{\alpha} S^r_{(i_1,i_2,\ldots,i_\alpha)}\right]\\
\overset{(a)}{\geq}&\mathbb{E}\left[\max_{1\leq i_2\leq n}\cdots\max_{1\leq i_\alpha\leq n}\sum_{r=1}^{\alpha} S^r_{(i_1^*,i_2,\ldots,i_\alpha)}\right]\\
=&\mathbb{E}\left[\max_{1\leq i_2\leq n}\cdots\max_{1\leq i_\alpha\leq n}\left(\sum_{r=2}^{\alpha} S^r_{(i_1^*,i_2,\ldots,i_\alpha)} + S^1_{(i_1^*,i_2,\ldots,i_\alpha)}\right)\right].
\end{aligned} \tag{21}$$

Since the choice of i_1^* is only sub-optimal, the inequality (a) in Equation (21) should be strict inequality. Notice that according to Equation (20), for any values of $i_2,i_3,\ldots,i_\alpha$, we have

$$S^1_{(i_1^*,i_2,\ldots,i_\alpha)} = \max_{1\leq i_1\leq n} S^1_{(i_1,i_2,\ldots,i_\alpha)},$$

which, combining Equation (17) and the fact that $\{S_i^r\}$ are i.i.d. random variables, yields

$$\begin{aligned}
\mathbb{E}\left[S^1_{(i_1^*,i_2,\ldots,i_\alpha)}\right] =&\mathbb{E}\left[\max_{1\leq i_1\leq n} S^1_{(i_1,i_2,\ldots,i_\alpha)}\right]\\
=&\mathbb{E}\left[\max_{1\leq i\leq n} S_i^1\right]\\
=&\mathbb{E}\left[T(n,K)\right].
\end{aligned} \tag{22}$$

As a second step, for any values of $i_3,i_4,\ldots,i_\alpha$, let us define i_2^* as

$$i_2^*(i_1^*,i_3,\ldots,i_\alpha) = \arg\max_{1\leq i_2\leq n} S^2_{(i_1^*,i_2,\ldots,i_\alpha)}.$$

Then similarly as Equation (21), by fixing i_2 to be i_2^*, we can obtain

$$\begin{aligned}
&\mathbb{E}\big[T(n^\alpha,\alpha K)\big]\\
>&\mathbb{E}\Bigg[\max_{1\leq i_3\leq n}\cdots\max_{1\leq i_\alpha\leq n}\\
&\left(\sum_{r=3}^{\alpha} S^r_{(i_1^*,i_2^*,\ldots,i_\alpha)} + S^1_{(i_1^*,i_2^*,\ldots,i_\alpha)} + S^2_{(i_1^*,i_2^*,\ldots,i_\alpha)}\right)\Bigg].
\end{aligned}$$

Also, for any values of $i_3,i_4\ldots,i_\alpha$, we have

$$\begin{aligned}
\mathbb{E}\left[S^2_{(i_1^*,i_2^*,\ldots,i_\alpha)}\right] =&\mathbb{E}\left[\max_{1\leq i_2\leq n} S^1_{(i_1^*,i_2,\ldots,i_\alpha)}\right]\\
=&\mathbb{E}\left[T(n,K)\right].
\end{aligned} \tag{23}$$

By defining $i_3^*,\ldots,i_\alpha^*$ in a similar way

$$\begin{aligned}
&i_u^*(i_1^*,\ldots,i_{u-1}^*,i_{u+1},\ldots,i_\alpha)\\
&=\arg\max_{1\leq i_u\leq n} S^u_{(i_1^*,\ldots,i_{u-1}^*,i_u,\ldots,i_\alpha)}
\end{aligned}$$

and iterating the above step, we can get

$$\begin{aligned}
&\mathbb{E}\big[T(n^\alpha,\alpha K)\big]\\
>&\mathbb{E}\left[S^1_{(i_1^*,i_2^*,\ldots,i_\alpha^*)} + S^2_{(i_1^*,i_2^*,\ldots,i_\alpha^*)} + \ldots + S^\alpha_{(i_1^*,i_2^*,\ldots,i_\alpha^*)}\right]\\
\overset{(b)}{=}&\sum_{r=1}^{\alpha}\mathbb{E}\left[S^r_{(i_1^*,i_2^*,\ldots,i_\alpha^*)}\right]\\
\overset{(c)}{=}&\alpha\mathbb{E}\left[T(n,K)\right].
\end{aligned} \tag{24}$$

Equation (b) follows from the fact that $\{S^r_{(i_1^*,i_2^*,\ldots,i_\alpha^*)}\}_{1\leq r\leq\alpha}$ are independent random variables and equation (c) follows from Equation (22), (23) and iterated steps. By combining Equation (1) and Equation (24), we have

$$\eta(n^\alpha,\alpha K) = \frac{\alpha K}{\mathbb{E}\left[T(n^\alpha,\alpha K)\right]} < \frac{\alpha K}{\alpha\mathbb{E}\left[T(n,K)\right]} = \eta(n,K),$$

which completes the proof. $\square$

7. CONCLUSION

In this paper, we characterize the throughput of a broadcast network using rateless codes. The broadcast channels are modeled by Markov modulated packet erasure channels, where the packet can either be erased or successfully received and for each receiver the current channel state distribution depends on the channel states in previous l packet transmissions.

We first characterize the asymptotic throughput of the system when n approaches infinity for any values of coding block size K as a function of number of receiver n in an explicit form. We show that as long as K scales at least as fast as $\log n$, we can achieve a non-zero asymptotic throughput. Under the more restrictive assumption that the channel is memoryless ($l = 0$), we study the case when K and n are finite. We show that, by keeping the ratio $K/\log n$ to be a constant, the system throughput will converge to the asymptotic throughput in a decreasing manner as n grows. By the help of these results, we are able to give a lower bound on the maximum achievable throughput (maximum achievable rate), which is a function of K, n and erasure probability $1-\gamma$. In contrast to the state-of-the-art, we show that our bound is asymptotically tight when $K/\log n$ is fixed as n approaches infinity. Further, through numerical evaluations, we show that our bound is significantly better than existing result.

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