Battle of Opinions over Evolving Social Networks

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***Abstract*—Social networking environments provide major platforms for the discussion and formation of opinions in diverse areas including, but not limited to, political discourse, market trends, news and social movements. Often, these opinions are of a competing nature, e.g., radical vs. peaceful ideologies, correct information vs. misinformation, one technology vs. another. We study battles of such competing opinions over evolving social networks. The novelty of our model is that it captures the exposure and adoption dynamics of opinions that account for the preferential and random nature of exposure as well as the persuasion power and persistence of different opinions. We provide a complete characterization of the mean opinion dynamics over time as a function of the initial adoption, as well as the particular exposure, adoption and persistence dynamics. Our analysis, supported by case studies, reveals the key metrics that govern the spread of opinions and establishes the means to engineer the desired impact of an opinion in the presence of other competing opinions.**

I. Introduction

Social networks, whether face-to-face or digital, capture the connections and interactions between people on a wide range of platforms. They are a medium for the spread of diverse influences including opinion, information, innovation, riots, biological or computer viruses, and even obesity [1]. As such, social networks play a key role in shaping human behavior.

In this paper, we focus on understanding and combining two key aspects of social influence spread: (i) the *dynamically evolving nature of social interconnections* and (ii) the *existence of multiple competing influences* in a social medium. While each of these aspects has been studied thoroughly in isolation, there is a lack of understanding when these two aspects operate at comparable time scales.

A motivating example for our study is commonly observed *battles of opinions* on social platforms, e.g., Twitter, as a reaction to a piece of possibly controversial news. In these scenarios, the information spread and opinion formation typically start with a small set of initial nodes (representing users). Over time, followers of these initial nodes are exposed to the news and opinions to form an opinion of their own. The opinion that a node adopts is affected by the opinion of their point of contact. Once a node adopts an opinion, it joins the dynamically growing opinion subnetwork of nodes that have heard the news and formed an opinion. Such scenarios exist in several other platforms including personal blogs, customer reviews on retailer sites, etc.

Understanding the fate of such competing opinions over social networks demands new models that capture the spreading and adoption dynamics of different opinions over a common network platform. This motivates us in this work to model and study the spreading dynamics of multiple influences over a growing dynamic network.

We require our network model to capture several phenomena, such as a heavy-tailed degree distribution and short average distance, that are observed in many real-world social networks. The *degree* or connectivity of a node in a network is the number of its connections. Online social networks such as Twitter have been shown to have a heavy-tailed degree distribution [2].

This heavy-tailed degree distribution of scale-free networks necessitates a new random graph model other than the Erdös-Rényi (ER) random graph. In an ER graph, any two nodes are connected with a given probability independently of other connections in the network, and such a *random attachment* model gives rise to a Poisson degree distribution as the number of nodes increases [3], [4]. In [5], the authors propose the *preferential attachment* model as a mechanism that gives rise to a power-law degree distribution [6]. In the preferential attachment model, the probability that a node is connected to a given node is proportional to the degree of the given node.

Various hybrid models that mix preferential and random attachment have been studied in several scenarios of growing networks, social or otherwise (e.g., [7], [8], [9]). In these works, the authors show that networks evolving according to hybrid random-preferential attachment models exhibit a power-law degree distribution and other desirable properties that mimic social networks (e.g., short average distance, large clustering coefficients and positive degree correlation).

There is a rich history of research on the problem of evolving complex networks, including scale-free networks based on preferential and hybrid models of network growth (e.g., [10], [11] and references therein). In a different thread of work, researchers have been investigating influence spread over social or other networks. At their core, these works aim to capture the dynamics and the limiting behavior of various types of influence, e.g., opinion, viruses, etc., over predominantly static networks (e.g., [12], [13], [14]). However, to the best of our knowledge these topics have been studied individually.

In addition, there is very little work that concentrates on influence propagation specifically on scale-free networks. In [15] and [16], the authors study the spread of a single virus in a static network generated according to the preferential attachment model. However, they do not seek to characterize the time evolution of the influence spread; their focus is on

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The work has in part been funded by: a grant from the Defense Thrust Reduction Agency (DTRA) HDTRA1-14-1-0058; the Army Research Office under grant W911NF-15-1-0277; and the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (2017R1D1A1B03029786).

conditions that give rise to a persistent epidemic. In [17], the authors provide simulation results on the spread of opinions on static scale-free networks.

The spread of multiple competing influences has been studied to a lesser degree of extent as well. In [18], the authors study the spread of two viruses over an arbitrary undirected static network using an SIS (susceptible infected susceptible) epidemic spread model. In [19], the authors focus on the spread of conflicting information over static complex networks.

In this paper, we capture the following phenomena: the preferential vs. random nature of attachment of newcomer nodes; the varying power of different types of influence in persuading newcomers to adopt their type; varying responsiveness of newcomers to adopt different influences, and finite vs. infinite lifetimes of nodes in the network. Our main contributions to this end can be summarized as:

- We develop a new model to rigorously characterize the spread of multiple competing influences over evolving social networks. We perform a discrete-time analysis of the mean system dynamics using linear time-varying system theory.
- We provide a continuous-time approximation to the discrete-time model that enables further insights. Through simulation studies and analytical arguments, we verify the closeness of our continuous-time approximation to the discrete-time exact solution.
- We translate our analytical results into qualitative insights on the characteristics and essential dynamics of important instances of the problem. These investigations reveal the impact of different attachment and adoption dynamics on the transient and limiting behavior of influence spread.

II. Network Evolution and Influence Propagation Model

In this paper, we study the propagation of multiple competing influences over a dynamically evolving and expanding network. To that end, we propose a model where multiple types of influences interact with each other as the underlying network expands with newcomer nodes and evolves as existing nodes leave the network. This model not only captures the popularity or prominence of the existing nodes as measured by their number of connections or degree (as in preferential attachment models), but also the possible differences in the persuasion power of the influences themselves, which is typically determined by the quality of a product or the strength of an opinion.

A. Network Evolution: Exposure to Opinions

The network evolution starts at time t_0 with $N_0 > 0$ initial nodes and a total degree of $D_0 > 0$. We use $N_{tot}[t]$ and $D_{tot}[t]$ to denote the total number of nodes and total degree at time t, respectively. At the end of each discrete-time period $t \in \{t_0+1, t_0+2, t_0+3, \ldots\}$, a new node arrives[1] and connects to one of the existing nodes in the network. We refer to the node to which the newcomer node connects as the *parent node*.

Our current model accounts for a single parent node for each newcomer node. While this assumption is certainly limiting, it still allows our model to capture many real-life scenarios where it is possible to identify a most influential existing node for each newcomer node. One example is singling out the node of *first exposure* as the parent node.

An important factor in determining which one of the existing nodes will be the source of exposure to the newcomer nodes is the visibility of the existing nodes as measured in terms of their connectivity. In the Twitter example, the higher the connectivity of a user, the more likely it is that the next user will hear the news from that particular user. Likewise, for customer reviews a higher number of helpful tags adds to the visibility of a particular review, and Google PageRank determines the visibility of personal blogs and other sites based on the hits they have received so far. In all examples, it is also possible that the next exposure will happen through a randomly selected, rather unassuming node.

In order to capture these dual connection dynamics we adopt a *hybrid* connection model composed of *random* and *preferential attachment*. Each newcomer node chooses either the random attachment mode with probability $q \in [0, 1]$ or the preferential attachment mode with probability $(1 - q)$ independently from the choices of the previous nodes. We refer to the probability q as the *attachment parameter*. In the random attachment mode, the newcomer node attaches to an existing node selected uniformly at random, i.e., each node in the network is chosen with equal probability $1/N_{tot}[t]$. In the preferential attachment mode, each node in the network is chosen with a probability that is proportional to its degree, i.e., if a particular node has degree d then it is chosen with probability $d/D_{tot}[t]$.

Note that newcomer nodes do not have an opinion when they first join the network. We model the connection making process to be independent of the opinions of the existing nodes. We say that user A is connected to user B if user A heard the news from node B or read the opinion of user B. Once exposed to the opinion of user B, user A will form an opinion based on her own personal and cognitive biases in addition to the influence of user B. The connection between users A and B will remain regardless of whether they share the same opinion or not.

B. Influence Propagation: Adoption of Opinion/Color

There are M different influences labeled $1, \ldots, M$ propagating in the network. Influence can refer to a wide range of things including opinions, ideas, innovations or products. In the sequel, we will use the word *color* when referring to these influences. Each node adopts only one out of M colors (hence the name competing influences) at the time it joins the network and does not change its color once adopted.

We assume that a newcomer node connects to the network according to the hybrid attachment model described in Section II-A independently of the colors of the existing nodes. This presumes that attachments are governed in part

[1] The model can be readily extended to the case where newcomer nodes arrive at possibly random times $\{T_1, T_2, \ldots\}$ with independent inter-arrival times. In that case, all our results still hold when the network is sampled right after the arrival of a new node.

by the random behavior of the newcomers and in part by the prominence of the existing nodes, but *not* by the adopted colors of the existing nodes. This assumption is justified in many scenarios where the colors of the existing nodes are not discernable by the newcomer node at the time of first connection, but their prominence is readily observable by the newcomer through their number of connections.

Once a newcomer connects to a parent node, it becomes receptive to the influence. The newcomer node is not restricted to adopt the same color as its parent node. The parent node's influence only determines the likelihood of the newcomer node adopting each color, e.g., the node can be more likely to adopt the same color as its parent. In particular, if the parent has color $j \in \{1, \dots, M\}$, then the newcomer node adopts color $i \in \{1, \dots, M\}$ with probability p_{ij}, i.e.,

$$p_{ij} = \mathbb{P}(\text{Node adopts color } i | \text{Parent node has color } j),$$

where $0 \leq p_{ij} \leq 1$ for all i and j, and $\sum_i p_{ij} = 1$ for each j. The set of adoption parameters $\{p_{ij}\}$ captures the *persuasion power* of different types of influences. Depending on the type of influence, these parameters may reflect the strength of an opinion or inherent quality of a product.

Although each node adopts a single color, this model can also encompass scenarios where newcomer nodes may adopt *zero* or *multiple* colors. Examples include the newcomer node not subscribing to any of the existing opinions, not buying any product, or buying multiple products. In these cases, a new color is assigned to these choices.

We use $N_i[t]$ and $D_i[t]$ to denote the number and the total degree of nodes of color i at time t, where $\sum_{i=1}^{M} N_i[t] = N_{tot}[t]$ and $\sum_{i=1}^{M} D_i[t] = D_{tot}[t]$. In order to facilitate a more compact presentation, we define the state vector

$$\mathbf{X}[t] \triangleq \begin{pmatrix} \mathbf{N}[t] \\ \mathbf{D}[t] \end{pmatrix} \in \mathbb{R}^{2M \times 1}, \tag{1}$$

in terms of the number of nodes of different color $\mathbf{N}[t] \triangleq (N_1[t], \dots, N_M[t])^T \in \mathbb{R}^{M \times 1}$ and degrees $\mathbf{D}[t] \triangleq (D_1[t], \dots, D_M[t])^T \in \mathbb{R}^{M \times 1}$. The initial state of the network at time t_0 is given by $\mathbf{X}[t_0] = \mathbf{X}_0$.

C. Persistence of Nodes: Finite and Infinite Node Lifetime

The influence of a node may or may not be indefinite depending on the nature of the node or the social platform. Motivated by this observation, we capture a node's *persistence* through its lifetime. The *lifetime* of a node refers to the duration that the node remains in the network before its influence dies out. During its lifetime, each node is visible to newcomer nodes and actively participates in spreading its influence.

In our discussions, we distinguish between scenarios where nodes have finite or infinite lifetimes since they lead to fundamentally different influence spread dynamics. In the first scenario of networks with infinite lifetimes: all nodes remain indefinitely in the network once they join, and influence later newcomer nodes. In this scenario, the network size and influence grow indefinitely[2]. Examples for this scenario include online forums or product recommendation platforms where messages or recommendations of earlier users remain visible indefinitely.

In the second scenario, we assume that each node in the network (initial nodes as well as nodes that join the network later) leaves the network after a finite time. With nodes in the network leaving at random times, the number of nodes in the network at any given time t no longer needs to be an increasing function of t. Such a finite lifetime scenario is a good fit for modeling online discussion forums where participating users join and leave, and thereby their influence is not indefinite.

D. Problem Statement

Our goal is to characterize the time evolution of the network state $\mathbf{X}[t]$ defined in (1) given the network evolution dynamics in Sections II-A and II-C and the influence propagation dynamics in Section II-B. In particular, we want to characterize the extent of spread for different influences (i.e., the fraction of nodes who adopt different colors) as a function of the initial makeup of the network (as captured by the initial network state $\mathbf{X}_0$), the newcomer nodes sensitivity to prominence (as captured by the connection parameter q which determines the ratio of preferential vs random attachment), the persuasion power of different opinions (as captured by the adoption parameters $\{p_{ij}\}$), and the persistence of nodes (as captured by the lifetime distribution of the nodes).

III. Influence Spread Dynamics with Infinite Node Lifetime

In this section, we provide analytical results that describe the mean dynamics of an expanding influence network with infinite node lifetime introduced in Section II. We first derive exact results based on the discrete-time (DT) model. In order to achieve further insights into the effect of the various system parameters on the evolution of the system, we develop and analyze an approximate continuous-time (CT) model. We use these results to reveal important network formation and influence dynamics in the case studies of the subsequent sections.

A. Discrete-Time Mean System Analysis

In this subsection, we provide an exact characterization of the mean behavior of the system dynamics in discrete-time by investigating the conditional mean drift of the system state $\mathbf{X}[t]$ defined as $\mathbb{E}[\mathbf{X}[t+1] - \mathbf{X}[t] \mid \mathbf{X}[t]]$. In particular, we obtain a linear system with time-varying coefficients to describe the mean system evolution. These coefficients provide valuable information concerning the impact of the hybrid attachment model and the persuasion power parameters on the spread and the degree distribution of different types of influences. Theorem 1 shows our main result regarding the nature of influence spread under such dynamics.

[2]For the network dynamics described in Section II-A, the total number of nodes and degrees at time t is given by $N_{tot}[t] = (t - t_0) + N_0$ and $D_{tot}[t] = 2(t - t_0) + D_0$, respectively.

Theorem 1 (Linear Time-Varying DT System Description and Solution)**.** *The one-step time evolution of the mean state for the network with infinite node lifetime described in Section II is governed by the following time-varying linear difference equation*

$$\mathbb{E}[\mathbf{X}[t+1]-\mathbf{X}[t]\mid\mathbf{X}[t]]=\mathbb{A}[t]\mathbf{X}[t], \tag{2}$$

for $t\in\{t_0,t_0+1,\ldots\}$ *and initial condition* $\mathbf{X}[t_0]=\mathbf{X}_0$. $\mathbb{A}[t]$ *is a* $2M\times 2M$ *matrix composed of four* $M\times M$ *constant submatrices* $\mathbb{A}_{ij}$, $N_{tot}[t]$ *and* $D_{tot}[t]$ *as follows:*

$$\mathbb{A}[t]=\left[\begin{array}{cc}\mathbb{A}_{11}/N_{tot}[t] & 2\mathbb{A}_{12}/D_{tot}[t]\\ \mathbb{A}_{21}/N_{tot}[t] & 2\mathbb{A}_{22}/D_{tot}[t]\end{array}\right], \tag{3}$$

where the entries of the constant submatrices are given by

$$\begin{aligned}
&[\mathbb{A}_{11}]_{i,j}=qp_{ij},\\
&[\mathbb{A}_{12}]_{i,j}=\frac{1}{2}(1-q)p_{ij},\\
&[\mathbb{A}_{21}]_{i,j}=\begin{cases}q(1+p_{ii}), & \text{if } i=j\\ qp_{ij}, & \text{if } i\neq j\end{cases}\\
&[\mathbb{A}_{22}]_{i,j}=\begin{cases}\frac{1}{2}(1-q)(1+p_{ii}), & \text{if } i=j\\ \frac{1}{2}(1-q)p_{ij}, & \text{if } i\neq j.\end{cases}
\end{aligned} \tag{4}$$

The mean state of the system at time t *is given by*

$$\mathbb{E}[\mathbf{X}[t]\mid\mathbf{X}_0]=\left(\prod_{s=t_0}^{t-1}(\mathbb{A}[s]+\mathbb{I})\right)\mathbf{X}_0, \tag{5}$$

where $\mathbb{I}$ *is the* $2M\times 2M$ *identity matrix. Since matrix multiplication is not necessarily commutative, we fix the order of multiplication in* (5) *as*

$$\prod_{s=t_0}^{t-1}(\mathbb{A}[s]+\mathbb{I})\triangleq(\mathbb{A}[t-1]+\mathbb{I})(\mathbb{A}[t-2]+\mathbb{I})\cdots(\mathbb{A}[t_0]+\mathbb{I}).$$

Proof. The proof is given in Appendix A. □

It is possible, and insightful, to derive a more explicit solution to the general equation governing the network evolution in (5) by imposing a restriction on the initial state of the system. We observe that the total degree in the network $D_{tot}[t]=2(t-t_0)+D_0$ approaches twice the number of nodes $N_{tot}[t]=(t-t_0)+N_0$ with increasing time t. If we impose the condition $D_0=2N_0$ from the onset to ensure $D_{tot}[t]=2N_{tot}[t]$ for all t, then we can write $\mathbb{A}[t]=\mathbb{A}/(t-t_0+N_0)$ where $\mathbb{A}$ is the constant matrix composed of the submatrices defined in (4) as follows:

$$\mathbb{A}=\left[\begin{array}{cc}\mathbb{A}_{11} & \mathbb{A}_{12}\\ \mathbb{A}_{21} & \mathbb{A}_{22}\end{array}\right].$$

The following corollary summarizes our results for this specific case.

Corollary 1. *Provided that* $D_0=2N_0$, *and that the matrix* $\mathbb{A}$ *is diagonalizable, the expected state of the network at time* t *with infinite node lifetime described in Section II is given by*

$$\mathbb{E}[\mathbf{X}[t]\mid\mathbf{X}_0]=\mathbb{V}\Lambda[t]\mathbb{V}^{-1}\mathbf{X}_0, \tag{6}$$

where $\Lambda[t]$ *is the* $2M\times 2M$ *diagonal matrix with entries*

$$[\Lambda[t]]_{i,i}=\exp\left(\sum_{s=t_0}^{t-1}\log\left(1+\frac{\lambda_i}{s-t_0+N_0}\right)\right) \tag{7}$$

and $\{\lambda_i\}_{i=1}^{2M}$ *and* $\mathbb{V}$ *are the eigenvalues and eigenvector matrix of* $\mathbb{A}$, *respectively.*

Proof. The result follows readily from (5) by replacing $\mathbb{A}[s]$ with $\mathbb{A}/(s-t_0+N_0)$ and $\mathbb{A}$ with $\mathbb{V}\operatorname{diag}\left(\{\lambda_i\}_{i=1}^{2M}\right)\mathbb{V}^{-1}$. □

B. Continuous-Time Approximation

In this subsection, we propose a continuous-time (CT) approximation to the mean evolution of the influence network. Throughout the paper, we use (t) instead of $[t]$ to distinguish continuous-time variables from their discrete-time counterparts. We introduce the shorthand notation $\mathbf{x}(t)\triangleq\mathbb{E}[\mathbf{X}(t)]$ to denote the CT approximation of the *mean* state vector. Next, we obtain a *heuristic* CT approximation for the evolution of the network by replacing the difference equation in (2) by a differential equation.

Definition 1 (Continuous-Time Approximation of the System State Evolution)**.** *The continuous-time evolution of the mean system state* $\mathbf{x}(t)$ *is described by the following time-varying linear differential equation:*

$$\frac{d\mathbf{x}(t)}{dt}=\mathbb{A}(t)\mathbf{x}(t),\ \text{for } t\geq t_0,\ \text{and } \mathbf{x}(t_0)=\mathbf{X}_0 \tag{8}$$

where $\mathbb{A}(t)$ *has the same form as* $\mathbb{A}[t]$ *defined in* (3)*.*

We derive an explicit solution to the system state evolution in (8) for the case that the initial state satisfies the constraint $D_0=2N_0$ as in Corollary 1. In this case, we note that $\mathbb{A}(s)$ commutes with $\mathbb{A}(t)$ for all values of s and t, i.e., $\mathbb{A}(s)\mathbb{A}(t)=\mathbb{A}(t)\mathbb{A}(s)$ for all s,t. The Magnus series [20] consists of a single term and yields the solution given in Corollary 2. Alternatively, we can show that (9) given below solves (8) by direct substitution.

Corollary 2. *When* $D_0=2N_0$, *the solution to* (8) *is given by*

$$\mathbf{x}(t)=\exp\left(\log\left(\frac{t-t_0+N_0}{N_0}\right)\mathbb{A}\right)\mathbf{X}_0. \tag{9}$$

For diagonalizable $\mathbb{A}$, *we can further reduce this solution by substituting the eigendecomposition* $\mathbb{A}=\mathbb{V}\operatorname{diag}\left(\{\lambda_i\}_{i=1}^{2M}\right)\mathbb{V}^{-1}$ *in the definition of the matrix exponential to obtain*

$$\mathbf{x}(t)=\mathbb{V}\operatorname{diag}\left(\left\{\left(\frac{t-t_0+N_0}{N_0}\right)^{\lambda_i}\right\}_{i=1}^{2M}\right)\mathbb{V}^{-1}\mathbf{X}_0. \tag{10}$$

Next we argue analytically that the CT approximate solution $\mathbf{x}(t)$ obtained in (10) is indeed a reasonable approximation of the DT exact solution $\mathbf{X}[t]$ obtained in (6). We start by noting that for small x, $\log(1+x)\approx x$. Hence, for large s, the $\log\left(1+\frac{\lambda_i}{s-t_0+N_0}\right)$ terms in (7) can be approximated by $\frac{\lambda_i}{s-t_0+N_0}$. Further approximating the sum of the harmonic terms by the corresponding integral results in the following approximation for the entries of the diagonal matrix:

$$\begin{aligned}&\exp\left(\sum_{s=t_0}^{t-1}\log\left(1+\frac{\lambda_i}{s-t_0+N_0}\right)\right)\\&\approx\exp\left(\lambda_i\sum_{s=t_0}^{t-1}\frac{1}{s-t_0+N_0}\right)\\&\approx\exp\left(\lambda_i\left(\log(t-t_0+N_0)-\log(N_0)\right)\right)\\&=\left(\frac{t-t_0+N_0}{N_0}\right)^{\lambda_i}\end{aligned}$$

With this approximation, (6) reduces to (10) as claimed. We also note that the diagonal terms corresponding to zero eigenvalues in the DT solution given in (6) and the CT approximate solution given in (10) are an exact match. Hence, only non-zero eigenvalues contribute to the difference between the two solutions.

Finally, we present a continuous-time arrival model under which the differential equation (8) holds exactly. To mimic the linear arrivals in the DT model (i.e., $N_{tot}[t]=(t-t_0)+N_0$ for $t\in\{t_0,t_0+1,\dots\}$), we suppose that the arrival process in the CT model is generated according to

$$\begin{aligned}&\mathbb{P}(\text{Exact 1 newcomer node arrives during } (k,k+\delta])=\delta,\\&\mathbb{P}(\text{No newcomer node arrives during } (k,k+\delta])=1-\delta,\end{aligned}$$

where $k\in\{t_0,t_0+1,\dots\}$ and $\delta\in(0,1]$. Then, the number of nodes under the CT model at time $t\in\{t_0,t_0+1,\dots\}$ is $N_{tot}(t)=(t-t_0)+N_0$ with probability 1, which is identical to that of the DT model. For this CT arrival model, we can extend the difference equation (2) in Theorem 1 to

$$\mathbb{E}[\mathbf{X}(t+\delta)-\mathbf{X}(t)|\mathbf{X}(t)]=\delta\mathbb{A}(t)\mathbf{X}(t), \tag{11}$$

for any $t\geq t_0$ and $\delta\in[0,1]$. Recalling that $\mathbf{x}(t)=\mathbb{E}[\mathbf{X}(t)]$, the expectation of the left-hand side of (11) reduces to

$$\begin{aligned}\mathbb{E}\big[\mathbb{E}[\mathbf{X}(t+\delta)-\mathbf{X}(t)|\mathbf{X}(t)]\big]&=\mathbb{E}[\mathbf{X}(t+\delta)-\mathbf{X}(t)]\\&=\mathbf{x}(t+\delta)-\mathbf{x}(t).\end{aligned}$$

Hence, the expectation of (11) yields

$$\mathbf{x}(t+\delta)-\mathbf{x}(t)=\delta\mathbb{A}(t)\mathbf{x}(t). \tag{12}$$

Since (12) holds for all $\delta\in(0,1]$, it follows that

$$\lim_{\delta\to 0}\frac{\mathbf{x}(t+\delta)-\mathbf{x}(t)}{\delta}=\mathbb{A}(t)\mathbf{x}(t),$$

which is the differential equation (8) in our CT approximation.

We have compared both DT and CT results and Monte Carlo simulations of our model for several sets of system parameters. Our results verify that the difference between the DT and CT evolutions is negligible. As part of these investigations, we show that actual simulation results are in line with theoretical results based on the CT approximation (cf. Figs. 1 and 2).

In the subsequent two sections, we proceed to translate these analytical results into insights on the characteristics and essential dynamics of important instances of the problem.

IV. Battle of Two Opinions With Infinite Node Lifetime

In this section, we present the detailed solution to the continuous-time approximation with two competing influences in a network with infinite node lifetimes. Binary systems arise in a vast number of real life scenarios that are based on adopting or rejecting a single opinion, belief, technology or product. The importance of studying the two influence case is not only due to its applicability to these scenarios. Its relative simplicity allows us to gain insights into the dynamics of influence propagation on evolving systems, which can be generalized to scenarios with larger number of influences.

We consider a scenario in which nodes in an evolving network adopt opinion 1 or opinion 2 as described in Section II. The system can be fully described in terms of the initial state $\mathbf{X}_0$, the attachment parameter q, and the two cross-adoption parameters p_{12} and p_{21}. The latter quantify the rate of *defection* from an opinion, i.e., the failure rate of an existing node to persuade newcomer nodes to subscribe to the same opinion as itself. We define $\widetilde{p}=p_{12}+p_{21}$ and exclude the degenerate case of $\widetilde{p}=0$ from our discussion. In this case, newcomer nodes adopt their parent node's opinion without fail.

We assume, without loss of generality, that the network evolution starts at time $t_0=0$. For the initial state $\mathbf{X}_0=(N_1(0),N_2(0),D_1(0),D_2(0))^T$, we impose the condition that $D_0=2N_0$ (i.e., $D_1(0)+D_2(0)=2(N_1(0)+N_2(0))$) in order to facilitate an algebraic solution. The following is the main result of this case study, which describes the evolution of mean adoption dynamics in terms of initial conditions as well as attachment and influence dynamics.

Theorem 2. *For the network evolution and influence propagation dynamics described above, the continuous-time approximation to the mean number of nodes $n_i(t)=\mathbb{E}[N_i(t)]$ adopting each opinion is given by*

$$\begin{aligned}n_1(t)&=\alpha_1(t+N_0)+\beta\left(\frac{t+N_0}{N_0}\right)^{\lambda}+\gamma,\\n_2(t)&=\alpha_2(t+N_0)-\beta\left(\frac{t+N_0}{N_0}\right)^{\lambda}-\gamma,\end{aligned} \tag{13}$$

where the coefficients α_i,β,γ and the exponent λ depend on the system parameters as follows:

$$\begin{aligned}&\lambda=1-\frac{1}{2}(1+q)\widetilde{p},\quad \alpha_1=\frac{p_{12}}{\widetilde{p}},\quad \alpha_2=\frac{p_{21}}{\widetilde{p}},\\&\beta=\frac{2(1-\widetilde{p})(p_{21}N_1(0)-p_{12}N_2(0))}{\widetilde{p}(2-(1+q)\widetilde{p})},\\&\gamma=\frac{(1-q)(p_{21}N_1(0)-p_{12}N_2(0))}{2-(1+q)\widetilde{p}}.\end{aligned}$$

Proof. The proof is given in Appendix B. □

Several observations can be made concerning the evolution of the mean number of nodes adopting each opinion.

Linear and Sublinear Terms in the Evolution: The first term in each expression indicates a *linear* growth of the mean number of nodes with time. The exponent that governs the second terms is common, and satisfies $\lambda\in[-1,1]$ for all system dynamics. The extreme case of $\lambda = 1$ is achieved only when $\widetilde{p} = 0$. Hence, the second term is *sublinear* and will eventually be dominated by the linear first term. It is also interesting to observe that λ can take negative values, in which case the contribution of the second terms vanish with t.

Long-Term Adoption Characteristics: In view of the previous observation, as long as the defection rate $\widetilde{p} > 0$, the long-term adoption of an opinion is dominated by the linearly increasing component of the evolution. In particular, the fractions of the two opinions in the network converge to $\alpha_1 = p_{12}/\widetilde{p}$ and $\alpha_2 = p_{21}/\widetilde{p}$, respectively. *Thus, the long-term market share of a product is not influenced by the attachment dynamics (as captured by q) or the initial number of the early adopters (as captured by $\mathbf{X}_0$), but solely by the persuasiveness of the opinions (as captured by cross-adoption probabilities p_{12} and p_{21}).*

Fig. 1 depicts the evolution of the theoretical CT approximation and results of Monte Carlo simulations which confirm this long-term behavior by showing that the fraction of two opinions converges to the same limit for different values of q. The datapoints are generated by running 100 simulation experiments with the given parameters on synthetically generated networks, and display the average.

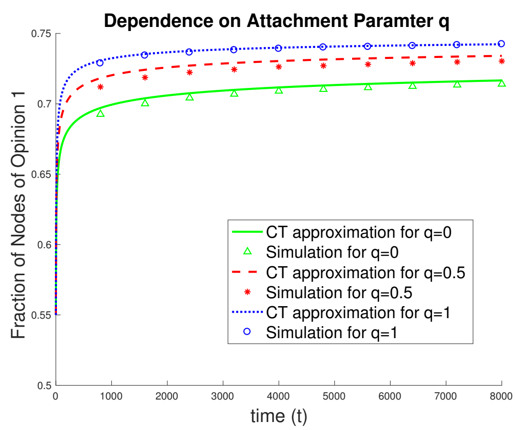


Fig. 1. The impact of the attachment parameter q on the mean fraction of nodes in an evolving network with two opinions. The graph depicts both the CT approximation given in Theorem 2 (continuous lines) and the average of 100 simulated experiments (individual datapoints). The attachment parameter varies as $q \in \{0, 0.5, 1\}$, while the cross-adoption parameters are fixed as $p_{12} = 0.3$ and $p_{21} = 0.1$ resulting in the limit $\alpha_1 = 0.75$ for the fraction of nodes adopting opinion 1. Note that the fraction of nodes adopting opinion 2 can be obtained by subtracting the fraction of nodes adopting opinion 1 from 1 and approaches $\alpha_2 = 0.25$ in the limit.

Impact of Attachment Model on the Evolution: Despite the dominance of the linear term in the long-term, the sublinear terms associated with the exponent λ and the coefficient β may have non-negligible *short-term* effects. In fact, such short-term characteristics may be of greater interest for many scenarios in which the influence spread occurs over a short/moderate lifetime. Here, we first observe that the exponent λ increases both with decreasing defection rate $\widetilde{p}$ and with decreasing randomness of attachment q. In other words, as the attachment model tends more towards pure preferential attachment, i.e., q decreases towards 0, the short-term effects are more pronounced in the exponent. Fig. 1 depicts this effect. The evolution curves with $q = 0$ corresponding to pure preferential attachment approach the limiting ratios α_1 and α_2 more slowly.

Impact of Initial Adopters on the Evolution: The coefficient β of the sublinear term depends on the composition of the early adopters as well as the cross-adoption probabilities. The dominant effect of the initial network composition on the evolution of the system is through this coefficient only. Fig. 2 depicts the evolution of the theoretical CT approximation and results of Monte Carlo simulations. First, we note how in accordance with the previous observations the long-term limits of α_1 and α_2 are unaffected by the initial network composition. We also observe that even when starting from an extreme initial condition, i.e., all initial nodes of a single opinion, the expected fraction of nodes of each opinion reaches an equilibrium in relatively short time.

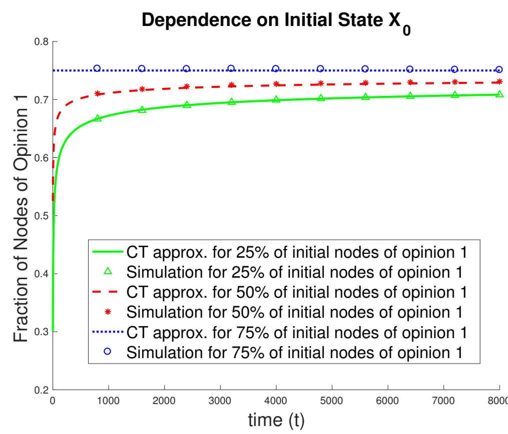


Fig. 2. The dependence of the mean fraction of nodes in an evolving network with two opinions on the initial state $\mathbf{X}_0$ of the network. The graph depicts both the CT approximation given in Theorem 2 (continuous lines) and the average of 100 simulated experiments (individual datapoints). Individual curves depict the fraction of nodes adopting opinion 1 over time starting with varying initial ratios $\{0.25, 0.5, 0.75\}$ of nodes of opinion 1. The attachment parameter is $q = 0.5$ and the cross-adoption parameters are $p_{12} = 0.3$ and $p_{21} = 0.1$ resulting in the limit $\alpha_1 = 0.75$ for the fraction of nodes adopting opinion 1. Note that the fraction of nodes adopting opinion 2 can be obtained by subtracting the fraction of nodes adopting opinion 1 from 1 and approaches $\alpha_2 = 0.25$ in the limit.

The above observations suggest an interesting *connection between attachment dynamics and the early spread of an influence*. In particular, the emergence of prominent (well-connected) members in a society as determined by the attachment dynamics allows the initial influence of the early adopters to survive longer. More specifically, as the q parameter decreases, the degree distribution has heavier tails, thereby indicating emergence of influential/prominent agents. The color of these prominent members will be shaped by the initial composition of the network, which, in turn, will sustain these early impacts for increasingly longer time frames depending on the value of λ. Yet, our model also reveals that the *long-term spread of two competing opinions is ultimately governed by their inherent strengths*.

V. Two Competing Technologies in a Network with Indifferent Populations and Infinite Node Lifetime

In this section, we study the dynamics of innovation spread in the case of two competing alternatives in an evolving network where nodes are allowed to remain *indifferent*, i.e., they adopt neither of the two technologies. We model a word-of-mouth marketing scenario in which newcomer nodes are exposed to the innovation only if their parent node has adopted one of the technologies. In that case, they can adopt one of the innovations (including the competitor of the technology adopted by their parent node) or they can remain indifferent. Indifferent nodes, on the other hand, play a special role in this model in which they do not expose newcomer nodes to either innovation. Any newcomer node that connects to an indifferent node remains indifferent with probability 1.

With the presence of indifferent nodes, not every node partakes in adopting the innovation. This is in contrast to the model in Section IV where each node actively participated in the battle of opinions. With indifferent nodes in the network, we are interested in both the individual number of adopters of each technology and the size of the entire market.

In the language of Section II, we have *three* colors: adopting one of the two technologies (labeled colors 1 and 2) and remaining indifferent (labeled color 3). Given the adoption dynamics described above the system can be fully described by the attachment parameter q and the adoption parameters p_{11}, p_{12}, p_{21} and p_{22}. (Note that $p_{13} = p_{23} = 0, p_{33} = 1$, $p_{31} = 1 - (p_{11} + p_{21})$ and $p_{32} = 1 - (p_{12} + p_{22})$.) As in the previous section, we assume, without loss of generality, that the network evolution starts at time $t_0 = 0$. We also assume that the initial number of nodes and initial total degree in the network satisfy $D_0 = 2N_0$.

Theorem 3. *For the network evolution and influence propagation dynamics described above, the continuous-time approximation to the mean number of nodes $n_i(t) = \mathbb{E}[N_i(t)]$ adopting each technology is given by*

$$\begin{aligned} n_1(t) &= \alpha_1 \left(\frac{t+N_0}{N_0}\right)^{\lambda_1} + \beta_1 \left(\frac{t+N_0}{N_0}\right)^{\lambda_2} + \gamma_1, \\ n_2(t) &= \alpha_2 \left(\frac{t+N_0}{N_0}\right)^{\lambda_1} + \beta_2 \left(\frac{t+N_0}{N_0}\right)^{\lambda_2} + \gamma_2, \end{aligned} \tag{14}$$

while the mean number of indifferent nodes is

$$n_3(t) = t + N_0 - n_1(t) - n_2(t).$$

The coefficients $\alpha_i \geq 0, \beta_i, \gamma_i$ are constants that depend on the system parameters $p_{11}, p_{12}, p_{21}, p_{22}, q$ and initial state $\mathbf{X}_0$. The exponents λ_1 and λ_2 are given by

$$\begin{aligned} \lambda_1 &= \frac{1}{2}(1-q) + \frac{1}{4}(1+q)(p_{11} + p_{22} + \Delta), \\ \lambda_2 &= \frac{1}{2}(1-q) + \frac{1}{4}(1+q)(p_{11} + p_{22} - \Delta), \end{aligned} \tag{15}$$

where $\Delta = \sqrt{(p_{11} - p_{22})^2 + 4p_{12}p_{21}}$. The exponents satisfy $\lambda_2 \leq \lambda_1 \leq 1$, and $\lambda_1 = 1$ if and only if

$$p_{11} + p_{21} = p_{12} + p_{22} = 1. \tag{16}$$

Proof. The derivation of (14) and (15) is similar to the proof of Theorem 2 given in Appendix B. Hence, we omit the details. To establish the range of the exponents, we note that

$$\begin{aligned} \Delta &= \sqrt{(p_{11} - p_{22})^2 + 4p_{12}p_{21}} \\ &\leq \sqrt{(p_{11} - p_{22})^2 + 4(1 - p_{22})(1 - p_{11})} \qquad (17) \\ &= \sqrt{((p_{11} + p_{22}) - 2)^2} = 2 - p_{11} - p_{22}. \qquad (18) \end{aligned}$$

Hence, we obtain the bound $p_{11} + p_{22} + \Delta \leq 2$ and conclude that $\lambda_1 \leq 1$. Note that (17) is met with equality if and only if (16) is satisfied. Therefore, $\lambda_1 = 1$ if and only if (16) holds. □

The dependence of the coefficients $\alpha_i, \beta_i, \gamma_i$ in Theorem 3 on the system parameters $\{p_{ij}\}, q$ and $\mathbf{X}_0$ is quite complex. For the case of $q = 1$ corresponding to pure random attachment, we have

$$\begin{aligned} \alpha_1 &= \frac{1}{2\Delta}\left((p_{11} - p_{22} + \Delta)N_1(0) + 2p_{12}N_2(0)\right), \\ \alpha_2 &= \frac{1}{2\Delta}\left(2p_{21}N_1(0) + (-p_{11} + p_{22} + \Delta)N_2(0)\right), \\ \beta_1 &= \frac{1}{2\Delta}\left((-p_{11} + p_{22} + \Delta)N_1(0) - 2p_{12}N_2(0)\right), \\ \beta_2 &= \frac{1}{2\Delta}\left((-2p_{21}N_1(0) + (p_{11} - p_{22} + \Delta)N_2(0)\right), \\ \gamma_1 &= \gamma_2 = 0. \end{aligned}$$

Several observations can be made concerning the evolution of the mean number of nodes adopting each technology and can be contrasted to the two-color case without indifferent nodes.

Sublinear Growth of the Market Size: We note that only the expression for the mean number of indifferent nodes $n_3(t)$ has a linear term. The mean number of nodes adopting one of the two active influences is governed by the sublinear t^{λ_1} term. According to Theorem 3, $\lambda_1 < 1$ unless nodes exposed to either form of innovation do not have the option of remaining indifferent. We exclude this case from the discussion below. As a result, the fraction of each active influence within the total network population tends to zero in the long term. Nevertheless, there are two important measures to be studied: the total number of nodes adopting a new technology and the fraction of each technology among these nodes, i.e., the market size and the market share.

Impact of Adoption Model on the Market Size: The size of the market is given by the total number of nodes adopting a new technology, i.e., $n_1(t) + n_2(t)$. While the market size is affected by all system parameters, the largest effect is due to the adoption parameters $\{p_{ij}\}$. In particular, the market size grows monotonically with growing sums $p_{11}+p_{21}$ and $p_{12} + p_{22}$, as these sums represent the probability that a node exposed to the innovation does adopt either form of it. We call this measure the *technology retention* probability. Fig. 3 depicts the growth of the market size with increasing technology retention probability.

Impact of Attachment Model on the Market Size: The growth of the market size is dominated by the $(\alpha_1 + \alpha_2)t^{\lambda_1}$ term. Hence, the largest impact of the attachment model on the market size, especially in the long term, is through the

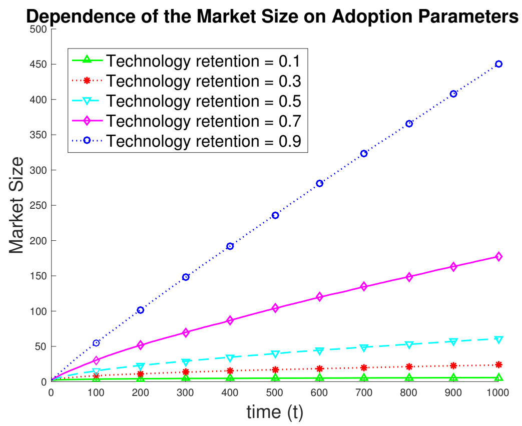


Fig. 3. The impact of the adoption parameters $\{p_{ij}\}$ on the market size in an evolving network with indifferent nodes. The technology retention probability $p_{11} + p_{21} = p_{12} + p_{22}$ varies as $\{0.1, 0.3, 0.5, 0.7, 0.9\}$. The fraction of nodes that opt for the same technology are 80% and 70%, respectively, for technologies 1 and 2. The attachment parameter is fixed as $q = 0.5$ and the network evolution starts with one node of each color. The graph shows the average of 100 simulated sample paths.

dependence of the exponent λ_1 on the attachment parameter q. In light of (18), $\lambda_1 = \frac{1}{4}(p_{11}+p_{22}+\Delta+2)+\frac{1}{4}q(p_{11}+p_{22}+\Delta-2)$ is a linearly decreasing function of q for all sets of adoption parameters $\{p_{ij}\}$. As a result, *the market size grows as the rate of random attachment q decreases*. A higher rate of preferential attachment allows individual nodes to establish higher prominence. Early technology adopters develop high degree, which attracts more of the newcomer nodes to one of the technologies, resulting in a larger market. Fig. 4 visualizes this effect of the attachment parameter q on the market size.

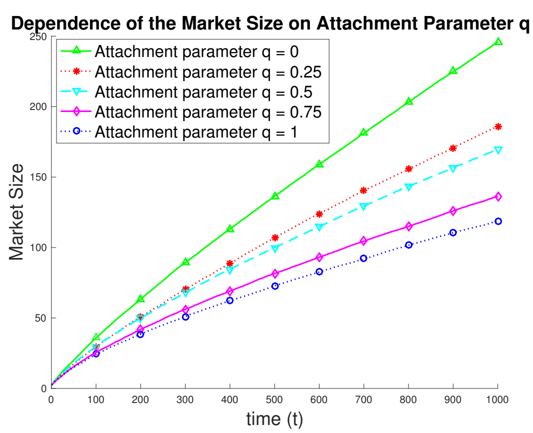


Fig. 4. The impact of the attachment parameter q on the market size in an evolving network with indifferent nodes. The attachment parameter varies as $q \in \{0, 0.25, 0.5, 0.75, 1\}$, while the adoption parameters are fixed as $p_{11} = 0.6, p_{21} = 0.1, p_{12} = 0.2$, and $p_{22} = 0.5$. The network evolution starts with one node of each color. The graph shows the average of 100 simulated sample paths.

Long-Term Market Share Characteristics: The long-term market share of each technology is determined by the coefficients α_1 and α_2 of the t^{λ_1} term in (14). These coefficients depend not only on the adoption parameters $\{p_{ij}\}$ but also on the attachment parameter q and the initial state of the network $\mathbf{X}_0$. This dependence is in apparent contrast to the previous case of two opinions presented in Section IV, where the leading coefficients α_1 and α_2 in (13) depended only on the adoption parameters. Nevertheless, the long-term market share of each product is not influenced by the attachment dynamics (as captured by q) nor the initial number of the early adopters (as captured by $\mathbf{X}_0$). In particular, the long-term fraction of technology 1 within the market is given as follows (product 2 occupies the remaining fraction of the market):

$$\frac{\alpha_1}{\alpha_1+\alpha_2} = \frac{p_{11} - 2p_{12} + \Delta - p_{22}}{2(p_{11} - p_{12} + p_{21} - p_{22})}.$$

Consequently, the attachment model and the preferences of the initial adopters have only short-term effects on the market share. In the long term, the effect of the adoption parameters dominates. Fig. 5 demonstrates how the effect of the initial network composition on the evolution of the market shares diminishes with time.

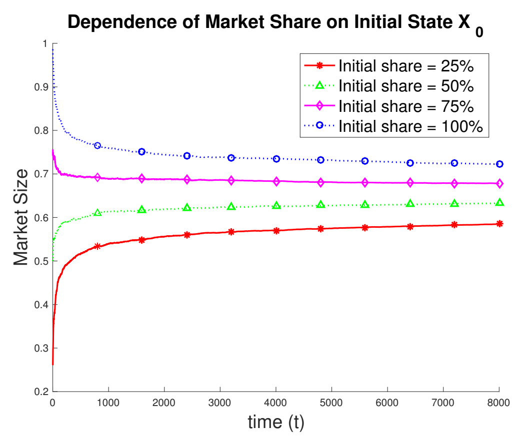


Fig. 5. The dependence of the market share of technology 1 on the initial network state. The initial market share of technology 1 varies as $\{0.25, 0.5, 0.75, 1\}$. The attachment parameter and the adoption parameters are fixed as $q = 0.5, p_{11} = 0.6, p_{21} = 0.1, p_{12} = 0.2$, and $p_{22} = 0.5$. The network evolution starts with ten nodes and the initial market size is 4. The graph shows the average of 100 simulated sample paths.

These observations reiterate the suggestion that the *long-term spread of two competing influences is ultimately governed by their inherent strength*. The attachment dynamics and the early adopters have only a secondary effect on the market size and market share.

VI. Influence Spread Dynamics with Finite Node Lifetime

In the previous sections (cf. Sections III-V), we have studied the dynamics of influence spread under the assumption that nodes remain in the network for an infinite time. That model captures scenarios where nodes are long-lived with respect to the influence spread and thus the spread occurs over *expanding* networks. In this section, we study an alternate setting in which the influence of participants has finite, albeit random, lifetimes, thereby resulting in influence spread over *evolving* networks. We first present a DT and a CT model that incorporate the lifetime dynamics into the network evolution. Then, we follow with our results on the limiting behavior and the stability of the resulting dynamic system, as well as numerical results in the case of two opinions competing over such dynamic networks.

A. Analysis of Mean System Dynamics

We model the lifetime of each node as a random variable whose statistics may depend on the adopted color of the node. In particular, we assume that a given node of color i leaves the network during the time slot $[t, t+1)$ (i.e., between the arrival of the t^{th} and $(t+1)^{st}$ node) with probability $\mu_i > 0$ independent of its connections and their actions. This is equivalent to assigning each node of color i a geometrically distributed lifetime with mean $1/\mu_i$. Note that, even though each node eventually leaves the network, the network does not go extinct due to the constant arrival of new nodes.

We know that, in the infinite lifetime scenario, preferential attachment gives rise to networks with heavy-tailed degree distributions (e.g., [6]). This results in the emergence of *prominent* nodes with significant attraction on newcomer nodes. In the finite lifetime scenario, this characteristic is diminished since the degree, hence the prominence, of a node is bounded by its finite and light-tailed lifetime. As a result, the system evolution is less sensitive to random versus preferential attachment dynamics. Therefore, in the remainder of this section, we focus on the case where attachment is purely random, i.e., $q = 1$.

Recall that in the presence of preferential attachment, the system state included the number as well as the degrees of nodes of each color (cf. Equation (1)). Now that we focus on random attachment only, the degrees of nodes no longer need to be part of the system state. We can therefore describe the system in terms of the number of nodes of each color only. In particular, we use $\mathbf{N}[t] = (N_1[t], \ldots, N_M[t])^T$ as the state vector with initial state $\mathbf{N}_0$.

Under our random attachment dynamics (cf. Section II-A), the newcomer node at time t attaches to an existing node of color j with probability $\frac{N_j[t]}{N_{tot}[t]}$. Then, under our influence dynamics (cf. Section II-B), it adopts color i with probability p_{ij}. Therefore, the number of nodes of color i increases by one during $[t, t+1)$ with probability $\sum_{j=1}^{M} p_{ij} \frac{N_j[t]}{N_{tot}[t]}$.

On the other hand, each one of the $N_i[t]$ nodes of color i can exit the network during $[t, t+1)$ independently with probability μ_i. Accordingly, the number of nodes of color i leaving in $[t, t+1)$ is the sum of $N_i[t]$ independent Bernoulli random variables with parameter μ_i, resulting in a Binomial distribution with parameters $N_i[t]$ and μ_i. Hence, the mean number of nodes of color i leaving in $[t, t+1)$ is given by $\mu_i N_i[t]$.

Combining such stochastic increase and decrease dynamics, we can express the evolution of the mean network state $\mathbb{E}[\mathbf{N}[t]]$ by the following system of non-linear difference equations:

$$\mathbb{E}[N_i[t+1] - N_i[t] \mid \mathbf{N}[t]] = \sum_{j=1}^{M} p_{ij} \frac{N_j[t]}{N_{tot}[t]} - \mu_i N_i[t],$$

for all $i = 1, \ldots, M$ and with the initial condition $\mathbf{N}[0] = \mathbf{N}_0$.

Following the same approach as in Section III-B, we next provide a CT approximation to this discrete-time system, where we use the shorthand notation $\mathbf{n}(t)$, $t \in \mathbb{R}_{\geq 0}$, to denote the CT approximation of the mean state vector $\mathbb{E}[\mathbf{N}[t]]$.

Definition 2 (Continuous-Time Approximation of the System State Evolution)**.** *The continuous-time evolution of the mean system state $\mathbf{n}(t)$ is described by the following time-varying non-linear differential equation: for each $i = 1, \ldots, M$,*

$$\frac{dn_i(t)}{dt} = \sum_{j=1}^{M} p_{ij} \frac{n_j(t)}{n_{tot}(t)} - \mu_i n_i(t), \quad \textit{for } t \geq 0, \tag{19}$$

and $\mathbf{n}(0) = \mathbf{N}_0$.

Note that the dynamics of this system differ in two fundamental ways from that of its infinite-lifetime counterpart given in (8). First, it includes a negative term that captures the departure dynamics. Second, $n_{tot}(t)$ is not a linear function of time, since it excludes nodes that have departed the system.

As such, the differential equation in (19) describes a non-linear system evolution in contrast to the linear time-varying nature of the system with infinite lifetime. This non-linear system does not lend itself to a closed-form solution that can describe the trajectory of the system state from any initial condition. Instead, we are interested in understanding the equilibrium and convergence characteristics of this system. In the following subsection, we undertake this task for the special case of $M = 2$ opinions. This case allows us to obtain an explicit expression for the equilibrium state and provide insights into the effects of the system parameters on the network evolution.

B. Battle of Two Opinions over Evolving Networks

In this subsection, we study the equilibrium and convergence characteristics of a system where two opinions spread over a network consisting of nodes with finite lifetimes. In particular, we consider the same scenario as in Section IV with the additional assumption that each node leaves the network after a finite time as described in Section VI-A.

The evolution of the system is governed by the attachment parameter q, the cross-adoption parameters p_{12}, p_{21}, and the *exit parameters* μ_1, μ_2. We assume that $p_{12} > 0$, $p_{21} > 0$, $\mu_1 > 0$, and $\mu_2 > 0$. In other words, we exclude the rather uninteresting case where each node adopts the same opinion as its parent, and we require that every node eventually leaves the network.

For our analysis, we focus on the case of purely random attachment as discussed in Section VI-A. In this case, the evolution of the CT approximate mean state $\mathbf{n}(t) = (n_1(t), n_2(t))^T$ is governed by the following equation:

$$\dot{\mathbf{n}}(t) = \frac{1}{n_{tot}(t)} \begin{bmatrix} 1 - p_{21} & p_{12} \\ p_{21} & 1 - p_{12} \end{bmatrix} \mathbf{n}(t) - \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix} \mathbf{n}(t), \tag{20}$$

where $n_{tot}(t) = n_1(t) + n_2(t)$. In Theorem 4, we present our results on the equilibrium and stability of this system.

Theorem 4 (Uniqueness of Equilibrium and Stability)**.** *For the dynamic system described by* (20) *we have:*

(i) *The dynamic system has a unique equilibrium state $\mathbf{n}^\star$ in $D = \{(n_1, n_2) \in \mathbb{R}^2 \backslash \{(0,0)\} : \ n_1 \geq 0, n_2 \geq 0\}$, given by:*

For $\mu_1 = \mu_2 = \mu$*:*

$$n_1^\star = \frac{p_{12}}{\mu(p_{12}+p_{21})}, \qquad n_2^\star = \frac{p_{21}}{\mu(p_{12}+p_{21})}.$$

For $\mu_1 \neq \mu_2$*:*

$$n_1^\star = \frac{1-\mu_2 n_{tot}^\star}{\mu_1-\mu_2}, \qquad n_2^\star = \frac{\mu_1 n_{tot}^\star - 1}{\mu_1-\mu_2}, \tag{21}$$

where

$$n_{tot}^\star = \frac{(1-p_{12})\mu_1 + (1-p_{21})\mu_2 + \sqrt{\Delta}}{2\mu_1\mu_2} \tag{22}$$

and

$$\Delta = [(1-p_{12})\mu_1 - (1-p_{21})\mu_2]^2 + 4p_{12}p_{21}\mu_1\mu_2. \tag{23}$$

(ii) *The equilibrium state* $\mathbf{n}^\star$ *is asymptotically stable, i.e., for any initial state* $\mathbf{n}(0)$ *in a small enough neighborhood of* $\mathbf{n}^\star$*, we have* $\lim_{t\to\infty}\mathbf{n}(t) = \mathbf{n}^\star$.

Proof. The proof is given in Appendix C. □

We note that the result in Theorem 4 when $\mu_1 = \mu_2 = \mu$ is in full agreement with the result of Theorem 2 in that the proportion of opinion adoptions at the equilibrium point $\mathbf{n}^\star$ is equal to the limiting proportions $\alpha_1 = p_{12}/\widetilde{p}$ and $\alpha_2 = p_{21}/\widetilde{p}$ in the infinite lifetime scenario. When $\mu_1 \neq \mu_2$, Theorem 4 reveals the non-trivial impact of asymmetric lifetimes on the spread and adoption of opinions.

Next, based on Theorem 4 and additional numerical investigations, we expand on the key characteristics associated with the transient and limiting behavior of the battle of two opinions over evolving networks.

Convergence to the Equilibrium: In Theorem 4, we have established the uniqueness of the equilibrium $\mathbf{n}^\star$ and the asymptotic stability of the system in (20) at that equilibrium. It is of interest to understand whether convergence can occur from a wide range around the equilibrium state. We have observed through numerical investigations that the system appears indeed to be *globally* asymptotically stable. A typical evolution that supports this conjecture is illustrated in Fig. 6, in which trajectories of the system state from various initial states clearly converge to the unique equilibrium.

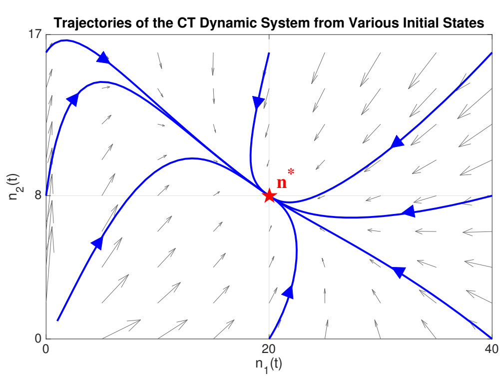


Fig. 6. The trajectories of the continuous-time dynamic system given in (20) starting from various initial states $\mathbf{n}(0)$. The system parameters are $p_{12} = 0.1, p_{21} = 0.2, \mu_1 = 0.03, \mu_2 = 0.05$ which corresponds to the equilibrium state $\mathbf{n}^\star = (20, 8)^T$.

We have also performed extensive numerical investigations of the system evolution for the discrete-time stochastic model presented in Section VI-A. As a typical example, the mean discrete-time stochastic state trajectory, averaged over 1000 simulated sample paths in each case, is depicted in Fig. 7 for the same system parameters and from the same initial states as in Fig. 6. Not surprisingly, the random and discrete nature of the evolution in the discrete-time stochastic system causes more jagged trajectories compared to the smooth trajectories of the continuous-time deterministic system. However, it is also apparent that the trajectories are drawn towards the same equilibrium state $\mathbf{n}^\star$ with the same directional drifts.

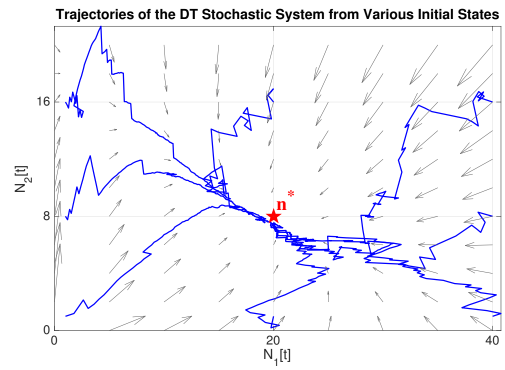
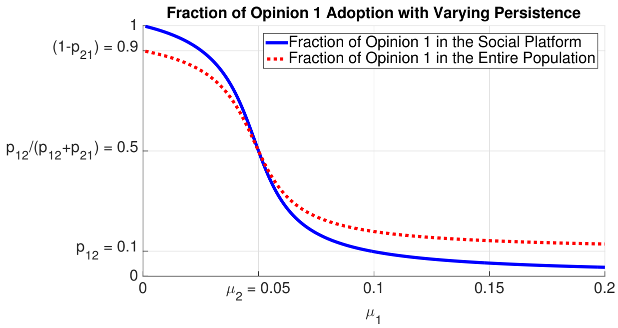


Fig. 7. The trajectories of the discrete-time stochastic system starting from various initial states $\mathbf{n}(0)$. The system parameters are $p_{12} = 0.1, p_{21} = 0.2, \mu_1 = 0.03, \mu_2 = 0.05$ which corresponds to the equilibrium state $\mathbf{n}^\star = (20, 8)^T$. The graph shows the average of 1000 simulated sample paths.

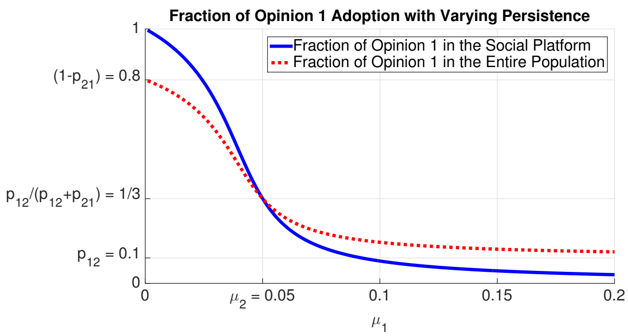
Impact of Lifetime on Opinion Adoption: Another set of interesting observations is concerned with the impact of lifetimes, which can also be interpreted as persistence of opinions, on the adoption dynamics. In Fig. 8, we demonstrate this effect in and outside of the social platform both under symmetric and asymmetric persuasion powers of the opinions (as measured by the adoption parameters p_{12} and p_{21}).

In particular, Fig. 8(a) investigates the characteristics of the equilibrium when both opinions have the same persuasion powers ($p_{11} = p_{22} = 0.9$) and where nodes with opinion 2 have a fixed exit probability of $\mu_2 = 0.05$. The solid line depicts the fraction of opinion 1 nodes that are active in the social platform (i.e., which have not yet exited) as the exit probability μ_1 for opinion 1 nodes varies from 0 to 0.2. First of all, we note that when $\mu_1 = \mu_2 = 0.05$, the opinion 1 and opinion 2 are equally represented in the social network. This is to-be-expected since all parameters are symmetric. But, when μ_1 is decreased slightly below μ_2, thereby assuming opinion 1 nodes are slightly more *persistent* than opinion 2 nodes, the fraction of opinion 1 nodes in the social network increases drastically. This nonlinear impact is due to the fact that by staying longer in the social platform, opinion 1 nodes increase their chances of influencing the newcomers and shift the balance towards their favor.

The solid line in Fig. 8(b) shows that the same impact can be effective even when the persuasion powers of the two opinions are asymmetric. In that setting, the intrinsic persuasion power of opinion 1 is taken to be smaller than that of opinion 2, i.e., $p_{11} = 0.8 < p_{22} = 0.9$. In this case, when the lifetimes



(a) $p_{12} = p_{21} = 0.1, \mu_2 = 0.05$



(b) $p_{12} = 0.1, p_{21} = 0.2, \mu_2 = 0.05$

Fig. 8. Fraction of nodes of opinion 1 within the social platform and in the entire population as a function of the varying exit parameter μ_1. The datapoints are obtained from Theorem 4.

are symmetric, opinion 1 is adopted by only $1/3$ of the social platform. Yet, reducing μ_1 below μ_2, opinion 1 can increase its share to any level up to 1. These investigations reveal the fact that *persistence can trump quality* in the battle of opinions.

Discrepancy of Representation in the Social Platform and Entire Population: Another interesting phenomenon is also illustrated in Fig. 8 that is concerned with the accuracy of representation in the social platform. The dashed lines in Figs. 8 (a) and (b) depict the fraction of opinion 1 nodes in the entire population. Therefore, this fraction is not restricted to the currently active participants, but includes those that have left the platform with their adopted opinions.

The figures show that, except when the lifetimes of the opinions are symmetric, the fraction of users in the social platform *exaggerates* the fraction of the opinions adopted in the entire population. This implies that the opinion distribution in a social platform is not necessarily a good representation of the adoption distribution of the entire population. In particular, we can see that as the persistence levels of opinions differ, the accuracy of the representations deviates.

VII. Conclusion

In this paper, we have studied the spread of multiple competing influences over evolving social networks. Motivated by the process of discussion and opinion formation over social networking platforms, we have introduced a new analytical network model where influence propagation and network evolution occur simultaneously. Our model has allowed us to capture a range of exposure, adoption, and lifetime dynamics of multiple interacting opinions. In particular, we have accounted for both the preferential and random nature of exposure, different persuasion powers of different influences as well as different degrees of persistence of nodes under each influence. We have analytically characterized the evolution of the mean influence spread over time as a function of the initial adoption, the exposure and adoption dynamics, and the persistence of nodes.

Our analysis, supported by several case studies, has provided insights on the short-term and long-term impacts of various network dynamics. For example, our analysis has revealed that the persistence of nodes and the persuasion power of an influence have potent long term effects on the spread of an influence, while the effects of exposure and initial adoption patterns subside over time.

Our findings on the short-term and long-term impact of initial adopters, persuasion powers, persistences, and exposure dynamics can guide involved parties in allocating their limited budget to maximize their influence over a social network. Our results have also revealed counterintuitive phenomena in some scenarios. For example, in the technology adoption setting, we saw that it can be better for a company to lose a customer to its competitor than to not have the customer get either of the two competing products.

There are a number of future directions that can be pursued to further develop our understanding of influence spread over dynamic networks. The arrival dynamics can be enriched to allow newcomers to possess prior biases and random arrival times. The exposure dynamics can also be generalized to allow influence from multiple existing nodes. Nodes can also be allowed to change their opinion after initial adoption based on their neighbors' states. We hope that the model and analysis presented in this paper form a motivation and foundation for these potential research directions.

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Appendix A
Proof of Theorem 1

The number of nodes of color i increases by one when the newcomer node adopts color i, regardless of the color of the parent node. Thus, the mean change in the number of nodes of color i from time t to $t+1$ is captured by the following difference equation:

$$\begin{aligned}
&\mathbb{E}[N_i[t+1]-N_i[t] \mid \mathbf{X}[t]] \\
&= \sum_{j=1}^{M} \mathbb{P}\left(\begin{array}{c}\text{New node connects to color } j\\ \text{and adopts color } i\end{array} \mid \mathbf{X}[t]\right) \\
&= \sum_{j=1}^{M} p_{ij}\mathbb{P}\left(\text{New node connects to color } j \mid \mathbf{X}[t]\right) \\
&= \sum_{j=1}^{M} p_{ij}\left(q\frac{N_j[t]}{N_{tot}[t]} + (1-q)\frac{D_j[t]}{D_{tot}[t]}\right). \qquad (24)
\end{aligned}$$

Next, to quantify the mean change in the total degree of the nodes of color i, note that the total degree of the nodes of color i increases by 2 when the newcomer node connects to an existing node of color i and adopts color i, while it increases only by 1 when the newcomer node connects to a node of color i but adopts another color, or when the newcomer node connects to a node of another color and adopts color i. This translates into the following conditional mean drift expression:

$$\begin{aligned}
&\mathbb{E}[D_i[t+1]-D_i[t] \mid \mathbf{X}[t]] \\
&= 2\mathbb{P}\left(\begin{array}{c}\text{New node connects to color } i\\ \text{and adopts color } i\end{array} \mid \mathbf{X}[t]\right) \\
&\quad + \sum_{j\neq i} \mathbb{P}\left(\begin{array}{c}\text{New node connects to color } i\\ \text{and adopts color } j\end{array} \mid \mathbf{X}[t]\right) \\
&\quad + \sum_{j\neq i} \mathbb{P}\left(\begin{array}{c}\text{New node connects to color } j\\ \text{and adopts color } i\end{array} \mid \mathbf{X}[t]\right) \\
&= 2p_{ii}\left(q\frac{N_i[t]}{N_{tot}[t]} + (1-q)\frac{D_i[t]}{D_{tot}[t]}\right) \\
&\quad + \sum_{j\neq i} p_{ji}\left(q\frac{N_i[t]}{N_{tot}[t]} + (1-q)\frac{D_i[t]}{D_{tot}[t]}\right) \\
&\quad + \sum_{j\neq i} p_{ij}\left(q\frac{N_j[t]}{N_{tot}[t]} + (1-q)\frac{D_j[t]}{D_{tot}[t]}\right) \\
&= (1+p_{ii})\left(q\frac{N_i[t]}{N_{tot}[t]} + (1-q)\frac{D_i[t]}{D_{tot}[t]}\right) \\
&\quad + \sum_{j\neq i} p_{ij}\left(q\frac{N_j[t]}{N_{tot}[t]} + (1-q)\frac{D_j[t]}{D_{tot}[t]}\right), \qquad (25)
\end{aligned}$$

where the last equality follows from the fact that $\sum_{i=1}^{M} p_{ij} = 1$. Combining (24) and (25) yields (2).

To obtain the solution to equation (2), we proceed as follows:

$$\begin{aligned}
&\mathbb{E}[\mathbf{X}[t] \mid \mathbf{X}_0] - \mathbf{X}_0 \\
&= \sum_{s=t_0}^{t-1} \mathbb{E}[\mathbf{X}[s+1]-\mathbf{X}[s] \mid \mathbf{X}_0] \qquad (26) \\
&= \sum_{s=t_0}^{t-1} \mathbb{E}[\mathbb{E}[\mathbf{X}[s+1]-\mathbf{X}[s] \mid \mathbf{X}[s],\mathbf{X}_0] \mid \mathbf{X}_0] \qquad (27) \\
&= \sum_{s=t_0}^{t-1} \mathbb{A}[s]\mathbb{E}[\mathbf{X}[s] \mid \mathbf{X}_0]. \qquad (28)
\end{aligned}$$

We express the mean state at time t as a telescoping sum of one-step mean drifts in equation (26). Equation (27) follows from the law of total expectation. We obtain equation (28) by applying the drift expression obtained in (2) to the inner expectation.

Finally, we arrive at the iterative expression for the mean state by grouping terms together. The final result (5) is obtained by iterating the last equation below:

$$\begin{aligned}
\mathbb{E}[\mathbf{X}[t] \mid \mathbf{X}_0] &= \mathbf{X}_0 + \sum_{s=t_0}^{t-1} \mathbb{A}[s]\mathbb{E}[\mathbf{X}[s] \mid \mathbf{X}_0] \\
&= \mathbf{X}_0 + \sum_{s=t_0}^{t-2} \mathbb{A}[s]\mathbb{E}[\mathbf{X}[s] \mid \mathbf{X}_0] \\
&\quad + \mathbb{A}[t-1]\mathbb{E}[\mathbf{X}[t-1] \mid \mathbf{X}_0] \\
&= (\mathbb{I} + \mathbb{A}[t-1])\mathbb{E}[\mathbf{X}[t-1] \mid \mathbf{X}_0].
\end{aligned}$$

Appendix B
Proof of Theorem 2

Throughout this proof, we use the complement notation $\overline{c} \triangleq 1-c$ in order to facilitate a compact notation.

The continuous-time evolution of the mean state is given by

$$\frac{d}{dt}\mathbf{x}(t) = \frac{1}{t+N_0}\mathbb{A}\mathbf{x}(t), \text{ for } t \geq 0,$$

and $\mathbf{x}(0) = \mathbf{X}_0$, where $\mathbb{A}$ is the constant matrix given by

$$\mathbb{A} = \begin{bmatrix} q\overline{p}_{21} & qp_{12} & \frac{1}{2}\overline{q}\cdot\overline{p}_{21} & \frac{1}{2}\overline{q}p_{12} \\ qp_{21} & q\overline{p}_{12} & \frac{1}{2}\overline{q}p_{21} & \frac{1}{2}\overline{q}\cdot\overline{p}_{12} \\ q(1+\overline{p}_{21}) & qp_{12} & \frac{1}{2}\overline{q}(1+\overline{p}_{21}) & \frac{1}{2}\overline{q}p_{12} \\ qp_{21} & q(1+\overline{p}_{12}) & \frac{1}{2}\overline{q}p_{21} & \frac{1}{2}\overline{q}(1+\overline{p}_{12}) \end{bmatrix}.$$

The matrix $\mathbb{A}$ is diagonalizable and has eigenvalues $\{1, 0, 0, \lambda = 1 - \frac{1}{2}(1+q)\widetilde{p}\}$. After an eigendecomposition, we evaluate the expression in (10) to obtain the results.

Appendix C
Proof of Theorem 4

We first prove the existence and uniqueness of the equilibrium state. Then, we apply Lyapunov stability theory to prove the stability of the equilibrium state. The domain of interest is given by $D = \{(n_1, n_2) \in \mathbb{R}^2 \backslash \{(0,0)\} : \ n_1 \geq 0, n_2 \geq 0\}$ since the number of nodes of any opinion is a nonnegative number. In what follows, we introduce new variables $\sigma(t) = n_1(t) + n_2(t) > 0$ and $\sigma^\star = n_1^\star + n_2^\star$ to simplify notation.

Proof of (i): The equilibrium state satisfies $\dot{\mathbf{n}}|_{\mathbf{n}=\mathbf{n}^\star} = 0$, which translates into the following pair of equations:

$$\dot{n}_1|_{\mathbf{n}=\mathbf{n}^\star} = (1-p_{21})\frac{n_1^\star}{\sigma^\star} + p_{12}\frac{n_2^\star}{\sigma^\star} - \mu_1 n_1^\star = 0, \quad (29)$$

$$\dot{n}_2|_{\mathbf{n}=\mathbf{n}^\star} = p_{21}\frac{n_1^\star}{\sigma^\star} + (1-p_{12})\frac{n_2^\star}{\sigma^\star} - \mu_2 n_2^\star = 0. \quad (30)$$

Alternatively, we can replace (30) by the sum of (29) and (30), which yields

$$\dot{\sigma}|_{\mathbf{n}=\mathbf{n}^\star} = 1 - \mu_1 n_1^\star - \mu_2 n_2^\star = 0. \quad (31)$$

Since $\sigma^\star > 0$, it suffices to solve the following two equations:

$$(1-p_{12}-p_{21})n_1^\star + p_{12}\sigma^\star - \mu_1 n_1^\star \sigma^\star = 0, \quad (32)$$

$$1 - (\mu_1 - \mu_2)n_1^\star - \mu_2\sigma^\star = 0. \quad (33)$$

First, we consider the case when $\mu_1 = \mu_2 = \mu$. In this case, the total number of nodes at the equilibrium state can be easily determined as $\sigma^\star = n_1^\star + n_2^\star = 1/\mu$, since (33) reduces to $1 - \mu\sigma^\star = 0$. After substituting this value for $\sigma^\star$, (32) yields a linear equation in $n_1^\star$ with the unique solution

$$n_1^\star = \frac{p_{12}}{\mu(p_{12}+p_{21})}.$$

It follows that

$$n_2^\star = \sigma^\star - n_1^\star = \frac{p_{21}}{\mu(p_{12}+p_{21})},$$

which proves part (i) of Theorem 4 for this particular subcase.

Next, we consider the case $\mu_1 \neq \mu_2$, and assume that $\mu_1 > \mu_2$ without any loss of generality. From (33) and the definition of $\sigma^\star$, it follows that $n_1^\star$ and $n_2^\star$ can be expressed in terms of $\sigma^\star$ as follows:

$$n_1^\star(\sigma^\star) = \frac{1-\mu_2\sigma^\star}{\mu_1 - \mu_2}, \quad (34)$$

$$n_2^\star(\sigma^\star) = \sigma^\star - n_1^\star(\sigma^\star) = \frac{\mu_1\sigma^\star - 1}{\mu_1 - \mu_2}.$$

Hence, we need to solve for $\sigma^\star$ and show that the solution is unique.

Combining (32) and (34), we obtain the quadratic equation

$$\mu_1\mu_2(\sigma^\star)^2 - [(1-p_{12})\mu_1 + (1-p_{21})\mu_2]\sigma^\star + (1-p_{12}-p_{21}) = 0. \quad (35)$$

We note that the discriminant of this equation is given by

$$\begin{aligned}\Delta &= [(1-p_{12})\mu_1 + (1-p_{21})\mu_2]^2 \\ &\quad - 4(1-p_{12}-p_{21})\mu_1\mu_2 \\ &= [(1-p_{12})\mu_1 - (1-p_{21})\mu_2]^2 \\ &\quad + 4(1-p_{21})(1-p_{12})\mu_1\mu_2 - 4(1-p_{12}-p_{21})\mu_1\mu_2 \\ &= [(1-p_{12})\mu_1 - (1-p_{21})\mu_2]^2 + 4p_{12}p_{21}\mu_1\mu_2 \geq 0.\end{aligned}$$

Thus, (35) has always at least one real valued solution. We let the two solutions $\sigma_1^\star \geq \sigma_2^\star$ be

$$\sigma_1^\star = \frac{(1-p_{12})\mu_1 + (1-p_{21})\mu_2 + \sqrt{\Delta}}{2\mu_1\mu_2},$$

$$\sigma_2^\star = \frac{(1-p_{12})\mu_1 + (1-p_{21})\mu_2 - \sqrt{\Delta}}{2\mu_1\mu_2}.$$

This proves that one equilibrium state is indeed given by (21) where $n_{tot}^\star = \sigma_1^\star$. Note that we need to show that this solution fulfills the conditions

$$\sigma_1^\star = n_1^\star + n_2^\star > 0, \quad n_1^\star \geq 0, \quad n_2^\star \geq 0. \quad (36)$$

It is trivial to see that $\sigma_1^\star > 0$. In what follows, we will show that $n_1^\star(\sigma_1^\star) \geq 0$ and $n_2^\star(\sigma_1^\star) \geq 0$. In addition, we will also show that $n_1^\star(\sigma_2^\star)$ and $n_2^\star(\sigma_2^\star)$ cannot be both nonnegative. Hence, this proves that $n_1^\star(\sigma_1^\star)$ and $n_2^\star(\sigma_1^\star)$ is the unique equilibrium state that fulfills all the conditions in (36).

Towards this end, we consider the product

$$n_1^\star(\sigma^\star)n_2^\star(\sigma^\star) = \frac{(1-\mu_1\sigma^\star)(\mu_2\sigma^\star - 1)}{(\mu_1-\mu_2)^2}$$

as a function of $\sigma^\star$. The conditions in (36) require that the product $n_1^\star(\sigma^\star)n_2^\star(\sigma^\star)$ is nonnegative which is the case if and only if the value of $\sigma^\star$ satisfies

$$\frac{1}{\mu_1} \leq \sigma^\star \leq \frac{1}{\mu_2}. \quad (37)$$

In the following three claims, we establish that $\sigma_1^\star$ does indeed satisfy (37) while $\sigma_2^\star$ does not.

Claim 1. $\sigma_1^\star \geq \frac{1}{\mu_1}$.

Proof: We want to show that

$$\sigma_1^\star = \frac{(1-p_{12})\mu_1 + (1-p_{21})\mu_2 + \sqrt{\Delta}}{2\mu_1\mu_2} \geq \frac{1}{\mu_1}.$$

Since $\mu_1 > 0$ and $\mu_2 > 0$, this is equivalent to showing

$$\sqrt{\Delta} \geq (1+p_{21})\mu_2 - (1-p_{12})\mu_1. \quad (38)$$

If $(1+p_{21})\mu_2 < (1-p_{12})\mu_1$, we are done since the right-hand side of (38) is negative. If $(1+p_{21})\mu_2 \geq (1-p_{12})\mu_1$, we can safely square both sides of (38) to obtain

$$\Delta \geq [(1+p_{21})\mu_2 - (1-p_{12})\mu_1]^2. \tag{39}$$

After substituting $\Delta = [(1-p_{12})\mu_1 - (1-p_{21})\mu_2]^2 + 4p_{12}p_{21}\mu_1\mu_2$ and basic algebraic manipulations, (39) reduces to

$$4p_{21}(\mu_1 - \mu_2)\mu_2 \geq 0,$$

which is true under our assumption $\mu_1 > \mu_2$.

Claim 2. $\sigma_1^\star \leq \frac{1}{\mu_2}$.

Proof: We need to show that

$$\sigma_1^\star = \frac{(1-p_{12})\mu_1 + (1-p_{21})\mu_2 + \sqrt{\Delta}}{2\mu_1\mu_2} \leq \frac{1}{\mu_2},$$

which is equivalent to showing that

$$\sqrt{\Delta} \leq (1+p_{12})\mu_1 - (1-p_{21})\mu_2 \tag{40}$$

since $\mu_1 > 0$ and $\mu_2 > 0$.

Note that, under our assumption of $\mu_1 > \mu_2$, the right-hand side of (40) is always positive. Hence, we can safely square both sides of (40) to obtain

$$\Delta \leq [(1+p_{12})\mu_1 - (1-p_{21})\mu_2]^2. \tag{41}$$

Further algebraic manipulations of (41) yield

$$4p_{12}\mu_1(\mu_1 - \mu_2) \geq 0,$$

which is true due to our assumption $\mu_1 > \mu_2$.

Claim 3. $\sigma_2^\star \leq \frac{1}{\mu_1}$.

Proof: We want to show that

$$\sigma_2^\star = \frac{(1-p_{12})\mu_1 + (1-p_{21})\mu_2 - \sqrt{\Delta}}{2\mu_1\mu_2} \leq \frac{1}{\mu_1}.$$

Since $\mu_1 > 0$ and $\mu_2 > 0$, this is equivalent to showing

$$\sqrt{\Delta} \geq (1-p_{12})\mu_1 - (1+p_{21})\mu_2. \tag{42}$$

The validity of (42) can be easily shown following the steps in the proof of Claim 1.

This completes the proof of part (i) of Theorem 4.

Proof of (ii): Our proof is based on showing that the eigenvalues of the Jacobian evaluated at the equilibrium state $\mathbf{n}^\star$ have strictly negative real parts. To that end, we introduce the function $\mathbf{f}(\mathbf{n}) = (f_1(\mathbf{n}), f_2(\mathbf{n}))^T$ to describe the time evolution of the state $\dot{\mathbf{n}} = \mathbf{f}(\mathbf{n})$. In particular,

$$\begin{bmatrix} f_1(\mathbf{n}) \\ f_2(\mathbf{n}) \end{bmatrix} = \begin{bmatrix} (1-p_{21})\frac{n_1}{\sigma} + p_{12}\frac{n_2}{\sigma} - \mu_1 n_1 \\ p_{21}\frac{n_1}{\sigma} + (1-p_{12})\frac{n_2}{\sigma} - \mu_2 n_2 \end{bmatrix},$$

where we continue to use the shorthand notation $\sigma = n_1 + n_2$ introduced in part (i).

The Jacobian matrix $\mathbb{J}_{\mathbf{f}}(\mathbf{n}) = \frac{d\mathbf{f}}{d\mathbf{n}} = \left[\frac{\partial f_i}{\partial n_j}\right]_{i,j}$ is given by $\mathbb{J}_{\mathbf{f}}(\mathbf{n}) = \frac{1}{\sigma^2}\cdot$

$$\begin{bmatrix} (1-p_{12}-p_{21})n_2 - \mu_1\sigma^2 & -(1-p_{12}-p_{21})n_1 \\ -(1-p_{12}-p_{21})n_2 & (1-p_{12}-p_{21})n_1 - \mu_2\sigma^2 \end{bmatrix}.$$

We will show that the real part of the eigenvalues of $\mathbb{J}_{\mathbf{f}}(\mathbf{n}^\star)$, i.e., the Jacobian matrix evaluated at the equilibrium point $\mathbf{n}^\star$, is negative. This implies the asymptotic stability of $\mathbf{n}^\star$ for the nonlinear system [21].

The eigenvalues of $\mathbb{J}_{\mathbf{f}}(\mathbf{n}^\star)$ are the solutions to the characteristic equation $\det(\lambda\mathbb{I} - \mathbb{J}_{\mathbf{f}}(\mathbf{n}^\star)) = 0$ and are given by

$$\lambda_{1/2} = \frac{1}{2\sigma^\star}\left(1 - p_{12} - p_{21} - (\mu_1+\mu_2)\sigma^\star \pm \sqrt{\delta}\right),$$

where $\sigma^\star = n_1^\star + n_2^\star$ and

$$\begin{aligned}\delta = {} & (1-p_{12}-p_{21})^2 + (\mu_1-\mu_2)^2(\sigma^\star)^2 \\ & + 2(1-p_{12}-p_{21})(\mu_1-\mu_2)(n_1^\star - n_2^\star).\end{aligned} \tag{43}$$

In the case when $\mu_1 = \mu_2 = \mu$, the two eigenvalues become $\lambda_1 = -\mu$ and $\lambda_2 = -\mu(p_{12}+p_{21})$. Clearly, both of these eigenvalues are strictly negative, and therefore asymptotic stability of $\mathbf{n}^\star$ is established.

In the following, we study the case when $\mu_1 \neq \mu_2$. First, we note that the $\lambda_{1/2} \mp \sqrt{\delta}/2\sigma^\star$ portion of the eigenvalues is always negative valued. In particular,

$$\begin{aligned}\lambda_{1/2} \mp \frac{\sqrt{\delta}}{2\sigma^\star} &= \frac{1-p_{12}-p_{21}-(\mu_1+\mu_2)\sigma^\star}{2\sigma^\star} \\ &= \frac{1-p_{12}-p_{21}-(\mu_1+\mu_2)(n_1^\star+n_2^\star)}{2\sigma^\star} \\ &= \frac{1-p_{12}-p_{21}-\mu_1 n_1^\star - \mu_1 n_2^\star - \mu_2 n_1^\star - \mu_2 n_2^\star}{2\sigma^\star} \\ &= -\frac{p_{12}+p_{21}+\mu_1 n_2^\star + \mu_2 n_1^\star}{2\sigma^\star} < 0,\end{aligned} \tag{44}$$

where (44) follows from $1 - \mu_1 n_1^\star - \mu_2 n_2^\star = 0$ (cf. (31)).

Next, we consider the real part of the $\mp\sqrt{\delta}/2\sigma^\star$ term. When $\delta < 0$, this term contributes no real part and (44) establishes that $\Re(\lambda_i) < 0$ for $i = 1, 2$. When $\delta \geq 0$ (i.e., $\lambda_i \in \mathbb{R}$), it suffices to show that

$$p_{12} + p_{21} + \mu_1 n_2^\star + \mu_2 n_1^\star > \sqrt{\delta}. \tag{45}$$

Since both sides of (45) are nonnegative, we can square both sides of the equation and validate that

$$\begin{aligned}&(p_{12}+p_{21}+\mu_1 n_2^\star + \mu_2 n_1^\star)^2 - \delta \\ &\quad = 4\mu_1\mu_2(\sigma^\star)^2 - 4(1-p_{12}-p_{21}) \qquad (46)\\ &\quad > 4\mu_1\mu_2\left(\frac{(1-p_{12})\mu_1 + (1-p_{21})\mu_2}{2\mu_1\mu_2}\right)^2 \\ &\qquad - 4(1-p_{12}-p_{21}) \qquad (47)\\ &\quad = ((\mu_1-\mu_2) - (p_{12}\mu_1 - p_{21}\mu_2))^2 \\ &\qquad + 4p_{12}p_{21}\mu_1\mu_2 > 0, \qquad (48)\end{aligned}$$

where (46) follows from substituting δ given in (43) and using the identity in (31); (47) follows from lower-bounding $\sigma^\star$ in (22) using the fact that Δ in (23) is nonnegative; and (48) follows from algebraic manipulations.

This assures that real parts of all eigenvalues of $\mathbb{J}_{\mathbf{f}}(\mathbf{n}^\star)$ are strictly negative, and therefore the system is asymptotically stable at the equilibrium state $\mathbf{n}^\star$.