Battle of Opinions over Evolving Social Networks

Irem Koprulu
Department of ECE
The Ohio State University
Email: irem.koprulu@gmail.com

Yoora Kim
Department of Mathematics
University of Ulsan, South Korea
Email: yrkim@ulsan.ac.kr

Ness B. Shroff
Departments of ECE and CSE
The Ohio State University
Email: shroff@ece.osu.edu

***Abstract*—Social networking platforms are responsible for the discussion and formation of opinions in diverse areas including, but not limited to, political discourse, market trends, news and social movements. Often, these opinions are of a competing nature, e.g., radical vs. peaceful ideology, one technology vs. another. We study the battle of such competing opinions over evolving social networks. The novelty of our model is that it captures the exposure and adoption dynamics of opinions that account for the preferential and random nature of exposure as well as the persuasion power of different opinions. We provide a complete characterization of the mean opinion dynamics over time as a function of the initial adoption as well as the particular exposure and adoption dynamics. Our analysis, supported by case studies, reveals the key metrics that govern the spread of opinions and establishes the means to engineer the desired impact of an opinion in the presence of other competing opinions.**

I. Introduction

Social networks, whether face-to-face or digital, capture the connections and interactions between people on a wide range of platforms. They are a medium for the spread of diverse influences including opinion, information, innovation, riots, biological or computer viruses, and even obesity [1]. As such, social networks play a key role in shaping human behavior.

We are interested in understanding the principles and dynamics of *multiple competing* influences spreading over an evolving social network. As an example, we regularly see battles of opinions on social platforms, e.g., Twitter, as a reaction to some news. The information spread and opinion formation start with a set of initial nodes. Over time, followers of these initial nodes are exposed to the news and join the dynamically growing *opinion subnetwork*, the network of nodes that have heard the news and formed an opinion. Even though the underlying social network platform, here Twitter, might be considered static over a shorter time frame, the opinion subnetwork over which the new influence originates and spreads is dynamically growing.

Understanding the evolution of such competing opinions over social networks demands new models that capture the spreading and adoption dynamics of opinions over a common network platform. This motivates us to model and study the spreading dynamics of multiple influences over a growing dynamic network. We require our network model to capture several phenomena, such as a heavy-tailed degree distribution, that are observed in many real-world social networks.

The *degree* or connectivity of a node in a network is the number of its connections. Online social networks such as Twitter have been shown to have a heavy-tailed degree distribution [2]. One class of networks with heavy-tailed degree distribution is *scale-free networks* in which the fraction $P(d)$ of vertices with degree d is proportional to $d^{-\gamma}$, where γ is a constant. The *preferential attachment* model has been proposed as a mechanism that gives rise to a power-law degree distribution [3]. In this model, the probability that a node is connected to a given node is proportional to the degree of the given node. This is in contrast to the *random attachment* model where any two nodes are connected with a given probability independently of other connections in the network.

Various hybrid models that mix preferential and random attachment have been studied in several scenarios of growing networks, social or otherwise (e.g., [4], [5], [6]). In these works, the authors show that networks evolving according to hybrid random-preferential attachment models exhibit a power-law degree distribution and other desirable properties that mimic social networks (e.g., short average distance, large clustering coefficients and positive degree correlation).

There is a rich history of research on the problem of evolving complex networks (e.g., [7], [8]) as well as influence propagation on static networks (e.g., [9], [10], [11]). However, to the best of our knowledge these topics have always been studied individually. In addition, there is very little work that concentrates on influence propagation specifically on scale-free networks. In [12] and [13], the authors study the spread of a single virus in a static network generated according to the preferential attachment model. However, they do not seek to characterize the time evolution of the virus spread; their focus is on conditions that give rise to a persistent epidemic.

In this work, we propose a new model for influence spread over an evolving network. Our model captures the preferential versus random nature of attachment, the varying persuasiveness of different types of influence and the varying responsiveness of nodes to adopt different influences. In particular, we want to answer the following key questions:

- How do the initial acceptance and the persuasiveness of influences affect their evolution? If the source of influence has limited resources to control the initial acceptance and the persuasiveness, how should it distribute it?

The work has in part been funded by: QNRF fund NPRP 5-559-2-227, grants from the Army Research Office (ARO) MURI W911NF-12-1-0385 and W911NF-15-1-0277; a grant from the Defense Thrust Reduction Agency (DTRA) HDTRA1-14-1-0058; and by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2014R1A1A2057793).

- What is the impact of preferential versus random attachment dynamics on the influence spread? Is the influence spread sensitive or robust to such dynamics?
- If persuasion parameters are not known a priori, can we infer them based on the observed influence spread?

In order to answer these questions, we first analyze the mean system dynamics to obtain insights into the evolution of influence. We then translate our analytical results into insights on the characteristics and essential dynamics of important instances of the problem. These investigations reveal the impact of different attachment and adoption dynamics on the transient and limiting behavior of influence spread.

II. Network Evolution & Influence Spread Model

We propose a model where multiple influences interact with each other on a network that expands with newcomer nodes. Our model captures both the popularity or prominence of the existing nodes as measured by their degree and the differences in the persuasion power of the influences themselves.

A. Network Evolution: Exposure to Opinions

The network evolution starts at time zero with $N_0 > 0$ initial nodes of total degree $D_0 > 0$. At the end of each discrete-time period $t \in \{1, 2, \ldots\}$, a new node arrives[1] and connects to one of the existing nodes in the network. We use $N_{tot}[t]$ and $D_{tot}[t]$ to denote the total number of nodes and total degree at time t, respectively.

We refer to the node to which the newcomer node connects as the *parent node*. Our current model accounts for a single parent node for each newcomer node. While this assumption is certainly limiting, it still allows us to capture many real life scenarios where it is possible to identify a most influential existing node for each newcomer node. One example is singling out the node of *first exposure* as the parent node.

We consider two modes of attachment to capture both the random behavior of the newcomer nodes and the prominence of the existing nodes. Each newcomer node attaches either *randomly* with probability $q \in [0, 1]$ or *preferentially* with probability $(1 - q)$ independently from the attachment modes of the previous nodes. We refer to the probability q as the *attachment parameter*. In the *random attachment* mode, the newcomer node attaches to an existing node selected uniformly at random, i.e., each node in the network is chosen with equal probability $1/N_{tot}[t]$. In the *preferential attachment* mode, each node in the network is chosen with a probability that is proportional to its degree, i.e., if a particular node has degree d then it is chosen with probability $d/D_{tot}[t]$.

B. Influence Propagation: Adoption of Opinion/Color

We consider M different influences labeled $1, \ldots, M$. Influence can refer to a wide range of things including opinions, ideas, innovations or products. In the sequel, we will use the word *color* when referring to these influences. Each node adopts only one out of M colors at the time it joins the network and does not change its color once adopted.

We assume that the color of its parent node is revealed to the newcomer node only after attachment. This assumption is justified in many scenarios where the colors of the existing nodes are not discernable at the time of connection making, but their prominence is readily observable by the newcomer through their number of connections. Once the newcomer node connects to a parent node, it becomes receptive to the influence. The parent node's color determines the likelihood of the newcomer node adopting each color. In particular, if the parent has color $j \in \{1, \ldots, M\}$, then the newcomer node adopts color $i \in \{1, \ldots, M\}$ with probability p_{ij}, i.e.,

$$p_{ij} = \mathbb{P}(\text{Node adopts color } i | \text{Parent node has color } j),$$

where $0 \leq p_{ij} \leq 1$ for all i and j, and $\sum_i p_{ij} = 1$ for each j. The set of adoption parameters $\{p_{ij}\}$ captures the *persuasion power* of different types of influences. Depending on the type of influence, these parameters may reflect the strength of an opinion or inherent quality of a product.

We use $N_i[t]$ and $D_i[t]$ to denote the number and the total degree of nodes of color i at time t. In order to facilitate a more compact presentation, we define the state vector

$$\mathbf{X}[t] = (N_1[t], \ldots, N_M[t], D_1[t], \ldots, D_M[t])^T \tag{1}$$

with initial value $\mathbf{X}[0] = \mathbf{X}_0$.

III. Mean System Dynamics

In this section, we provide analytical results that describe the mean dynamics of the evolving influence network introduced in Section II. We first derive exact results based on a discrete-time (DT) arrival model. The form of the exact results, however, provides only a limited insight into the effect of the various system parameters on the system evolution. In order to achieve further insight, we develop and analyze an approximate continuous-time (CT) model.

A. Discrete-Time Mean System Analysis

In this subsection, we provide an exact characterization of the mean behavior of the system dynamics in discrete-time by investigating the conditional mean drift of the system state $\mathbf{X}[t]$ defined in (1). In particular, we obtain a linear system with time-varying coefficients to describe the mean system evolution and present its solution.

Theorem 1 (Linear Time-Varying DT System Description and Solution)**.** *The one-step time evolution of the mean network state described in Section II is governed by the following time-varying linear difference equation*

$$\mathbb{E}[\mathbf{X}[t+1] - \mathbf{X}[t] \mid \mathbf{X}[t]] = \mathbb{A}[t]\mathbf{X}[t] \text{ for } t \in \{0, 1, \ldots\} \tag{2}$$

with initial condition $\mathbf{X}[0] = \mathbf{X}_0$. $\mathbb{A}[t]$ is a $2M \times 2M$ matrix composed of four $M \times M$ constant submatrices $\mathbb{A}_{ij}$, $N_{tot}[t]$ and $D_{tot}[t]$ as follows:

$$\mathbb{A}[t] = \begin{bmatrix} \mathbb{A}_{11}/N_{tot}[t] & 2\mathbb{A}_{12}/D_{tot}[t] \\ \mathbb{A}_{21}/N_{tot}[t] & 2\mathbb{A}_{22}/D_{tot}[t] \end{bmatrix}, \tag{3}$$

[1]The model can be readily extended to the case where newcomer nodes arrive at possibly random times $\{T_1, T_2, \ldots\}$ with independent inter-arrival times. In that case, all our results still hold when the network is sampled right after the arrival of a newcomer node.

where the entries of the constant submatrices are given by

$$\begin{aligned}
[\mathbb{A}_{11}]_{i,j} &= qp_{ij}, \\
[\mathbb{A}_{12}]_{i,j} &= \frac{1}{2}(1-q)p_{ij}, \\
[\mathbb{A}_{21}]_{i,j} &= \begin{cases} q(1+p_{ii}), & \text{if } i=j \\ qp_{ij}, & \text{if } i\neq j \end{cases} \\
[\mathbb{A}_{22}]_{i,j} &= \begin{cases} \frac{1}{2}(1-q)(1+p_{ii}), & \text{if } i=j \\ \frac{1}{2}(1-q)p_{ij}, & \text{if } i\neq j. \end{cases}
\end{aligned} \tag{4}$$

The mean state of the system at time t is given by

$$\mathbb{E}[\mathbf{X}[t] \mid \mathbf{X}_0] = \left(\prod_{s=0}^{t-1}(\mathbb{A}[s]+\mathbb{I})\right)\mathbf{X}_0,$$

where $\mathbb{I}$ is the $2M \times 2M$ identity matrix.

Proof. The proof is given in [14]. □

B. Continuous-Time Approximation

In this subsection, we provide a heuristic continuous-time approximation to the mean evolution of the influence network. Throughout the paper, we use (t) instead of $[t]$ to distinguish continuous-time variables from their discrete-time counterparts. We introduce the short-hand notation $\mathbf{x}(t) \triangleq \mathbb{E}[\mathbf{X}(t)]$ to denote the CT approximation of the *mean* state vector. Next, we obtain a *heuristic* continuous-time approximation for the evolution of the network by replacing the difference equation in (2) by a differential equation.

Definition 1 (Continuous-Time Approximation of the System State Evolution)**.** *The continuous-time evolution of the mean system state $\mathbf{x}(t)$ is described by the following time-varying linear differential equation:*

$$\frac{d\mathbf{x}(t)}{dt} = \mathbb{A}(t)\mathbf{x}(t), \text{ for } t \geq 0, \text{ and } \mathbf{x}(0) = \mathbf{X}_0 \tag{5}$$

where $\mathbb{A}(t)$ has the same form as $\mathbb{A}[t]$ defined in (3)*.*

It is possible, and insightful, to derive an explicit solution to (5) by imposing a restriction on the initial state of the system. We observe that the total degree in the network $D_{tot}(t) = 2t + D_0$ approaches twice the number of nodes $N_{tot}(t) = t + N_0$ with increasing time t. If we impose the condition $D_0 = 2N_0$ from the onset to ensure $D_{tot}(t) = 2N_{tot}(t)$ for all t, then we can write $\mathbb{A}(t) = \mathbb{A}/(t+N_0)$ where $\mathbb{A}$ is the constant matrix composed of the submatrices defined in (4) as follows:

$$\mathbb{A} = \begin{bmatrix} \mathbb{A}_{11} & \mathbb{A}_{12} \\ \mathbb{A}_{21} & \mathbb{A}_{22} \end{bmatrix}. \tag{6}$$

In this case, we note that $\mathbb{A}(s)$ commutes with $\mathbb{A}(t)$ for all values of s and t, i.e., $\mathbb{A}(s)\mathbb{A}(t) = \mathbb{A}(t)\mathbb{A}(s)$ for all s, t. The Magnus series [15] consists of a single term and yields the solution given in Corollary 1.

Corollary 1. *When $D_0 = 2N_0$, the solution to* (5) *is given by*

$$\mathbf{x}(t) = \exp\left(\log\left(\frac{t+N_0}{N_0}\right)\mathbb{A}\right)\mathbf{X}_0.$$

For diagonalizable $\mathbb{A}$, we can further reduce (1) *by substituting the eigendecomposition $\mathbb{A} = \mathbb{V}\operatorname{diag}\left(\{\lambda_i\}_{i=1}^{2M}\right)\mathbb{V}^{-1}$ in the definition of the matrix exponential to obtain*

$$\mathbf{x}(t) = \mathbb{V}\operatorname{diag}\left(\left\{\left(\frac{t+N_0}{N_0}\right)^{\lambda_i}\right\}_{i=1}^{2M}\right)\mathbb{V}^{-1}\mathbf{X}_0.$$

In [14], we compare both DT and CT results and Monte Carlo simulations of our model for several sets of system parameters. Our results verify that the difference between the DT and CT evolutions is negligible.

In the subsequent two sections, we translate these analytical results into insights on the characteristics and essential dynamics of important instances of the problem. These investigations reveal the impact of different attachment and adoption dynamics on the transient and limiting behavior of influence spread.

IV. Battle of Two Opinions

Two opinion systems arise in a vast number of scenarios that are based on adopting or rejecting an opinion, belief, technology or product. The importance of studying the two influence case is not only due to its wide applicability though. Its relative simplicity allows us to gain insights into the dynamics of influence spread, which can be generalized to scenarios with larger number of influences. In this section, we present the detailed solution to the continuous-time approximation with two competing influences in the network.

We consider an evolving network in which nodes adopt opinion 1 or 2 as described in Section II. The system can be fully described in terms of the attachment parameter q, the cross-adoption parameters p_{12} and p_{21}, and the initial state $\mathbf{X}_0$. We impose the condition $D_1(0)+D_2(0) = 2(N_1(0)+N_2(0))$ for $\mathbf{X}_0$ in order to facilitate an algebraic solution. The cross-adoption parameters p_{12} and p_{21} quantify the rate of *defection* from an opinion, i.e., the failure rate of an existing node to persuade newcomer nodes to subscribe to the same opinion as itself. We define $\widetilde{p} = p_{12} + p_{21}$ and exclude the degenerate case of $\widetilde{p} = 0$ from our discussion. In this case, newcomer nodes adopt their parent node's opinion without fail.

The following main result of this case study describes the evolution of mean adoption dynamics in terms of initial conditions as well as attachment and influence dynamics.

Theorem 2. *For the network evolution and influence propagation dynamics described above, the continuous-time approximation to the mean number of nodes $n_i(t) = \mathbb{E}[N_i(t)]$ adopting each opinion is given by*

$$\begin{aligned}
n_1(t) &= \alpha_1(t+N_0) + \beta\left(\frac{t+N_0}{N_0}\right)^{\lambda} + \gamma, \\
n_2(t) &= \alpha_2(t+N_0) - \beta\left(\frac{t+N_0}{N_0}\right)^{\lambda} - \gamma,
\end{aligned} \tag{7}$$

where the coefficients α_i, β, γ and the exponent λ depend on the system parameters as follows:

$$\lambda = 1 - \frac{1}{2}(1+q)\widetilde{p}, \quad \alpha_1 = \frac{p_{12}}{\widetilde{p}}, \quad \alpha_2 = \frac{p_{21}}{\widetilde{p}},$$

$$\beta = \frac{2(1-\widetilde{p})(p_{21}N_1(0) - p_{12}N_2(0))}{\widetilde{p}(2-(1+q)\widetilde{p})},$$
$$\gamma = \frac{(1-q)(p_{21}N_1(0) - p_{12}N_2(0))}{2-(1+q)\widetilde{p}}.$$

Proof. The proof is given in [14]. □

Several observations can be made concerning the evolution of the mean number of nodes adopting each opinion.

Linear and Sublinear Terms in the Evolution: The first terms in (7) indicate a *linear* growth of the mean number of nodes with time. The exponent that governs the second terms is common, and satisfies $\lambda \in [-1, 1]$ with $\lambda = 1$ only when $\widetilde{p} = 0$. Hence, the second terms are *sublinear* and will eventually be dominated by the linear first terms. It is also interesting to observe that λ can take negative values, in which case the contribution of the second terms vanishes with t.

Long-Term Adoption Characteristics: As long as $\widetilde{p} \neq 0$, the linear terms in (7) dominate the long-term adoption of an opinion. The fractions of the two opinions in the network converge to $\alpha_1 = p_{12}/\widetilde{p}$ and $\alpha_2 = p_{21}/\widetilde{p}$. Thus, *the long-term fraction of an opinion is not influenced by the attachment dynamics (captured by q) or the initial number of the early adopters (captured by $\mathbf{X}_0$), but solely by the persuasiveness of the opinions (captured by p_{12} and p_{21}).* Fig. 1 shows how the fractions of two opinions converge to the same limit for different values of q and confirms this observation.

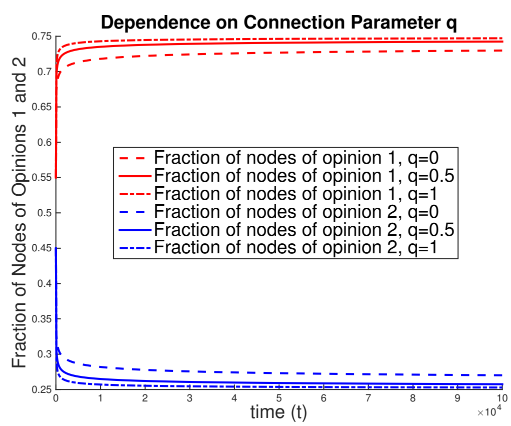

Fig. 1. The impact of a varying attachment parameter $q \in \{0, 0.5, 1\}$ on the mean fraction of opinions for $p_{12} = 0.3, p_{21} = 0.1$. The upper and lower set of curves depict the fraction of nodes adopting opinion 1 and 2, respectively.

Impact of Attachment Model on the Evolution: The sublinear terms in (7) associated with the exponent λ and coefficient β may have non-negligible *short-term* effects. Such short-term characteristics may be of greater interest for scenarios in which the influence spread occurs over a short/moderate lifetime. We observe that for a fixed defection rate $\widetilde{p}$ the exponent λ increases as the rate of random attachment q decreases. As the attachment model tends more towards pure preferential attachment, i.e., q decreases towards 0, the effects of the sublinear term are more pronounced. Fig. 1 shows how the evolution with pure preferential attachment, i.e., $q = 0$, approaches the limits α_1 and α_2 more slowly.

Impact of Initial Adopters on the Evolution: The coefficient β of the sublinear term depends on the composition of the early adopters as well as the cross-adoption probabilities. The effect of the initial network composition on the evolution of the system is through this coefficient only and is depicted in Fig. 2. First, we note how in accordance with the previous observations the long-term limits of α_1 and α_2 are unaffected by the initial network composition. We also observe that even starting from an extreme initial condition, i.e., all initial nodes of a single opinion, the expected fraction of nodes of each opinion reaches an equilibrium in relatively short time.

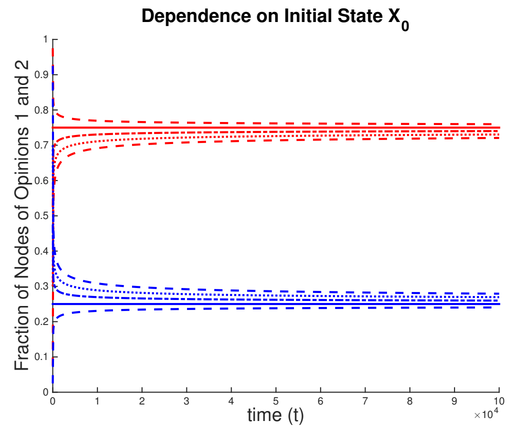

Fig. 2. The dependence of the mean evolution on the initial state $\mathbf{X}_0$, where the evolution starts with ratios $\{0, 0.25, 0.5, 0.75, 1\}$ of nodes of each opinion. The remaining parameters are $q = 0.5, p_{12} = 0.3, p_{21} = 0.1$. The upper and lower set of curves depict the fraction of nodes adopting opinion 1 and 2, respectively.

The above observations suggest an interesting *connection between attachment dynamics and the early spread of an influence*. In particular, the emergence of prominent (well-connected) members in the network as determined by the attachment dynamics allows the initial influence of the early adopters to survive longer. More specifically, as the attachment parameter q decreases, the degree distribution has heavier tails, thereby indicating emergence of influential/prominent nodes. The color of these prominent nodes will be shaped by the initial composition of the network, which, in turn, will sustain these early impacts for increasingly longer time frames depending on the value of λ. Yet, our model also reveals that the *long-term spread of two competing opinions is ultimately governed by their persuasion power.*

V. Two Competing Technologies in a Network with Indifferent Populations

In this section, we study the adoption dynamics of two competing technologies in an evolving network where nodes are allowed to remain *indifferent*, i.e., they adopt neither of the two technologies. We model a word-of-mouth marketing scenario in which newcomer nodes are exposed to the innovation only if their parent node has adopted one version of it. Once exposed, they can adopt one of the innovations (including the competitor of the technology adopted by the parent node) or they can remain indifferent. Indifferent nodes do not expose newcomer nodes to either technology. Any newcomer node that connects to an indifferent node remains indifferent with probability 1.

With the presence of indifferent nodes, not every node partakes in adopting the innovation. This is in contrast to the model in Section IV where each node actively participated in the battle of opinions. With indifferent nodes in the network, we are interested both in the individual number of adopters of each technology and the size of the entire market.

In the language of Section II, we have *three* colors: adopting one of the two technologies (labeled 1 and 2) and remaining indifferent (labeled 3). Given the adoption dynamics described above, the system can be fully described by the initial state $\mathbf{X}_0$, the attachment parameter q and the adoption parameters p_{11}, p_{12}, p_{21} and p_{22}. Note that $p_{13} = p_{23} = 0, p_{33} = 1$, $p_{31} = 1 - p_{11} - p_{21}$ and $p_{32} = 1 - p_{12} - p_{22}$. As in Section IV, we assume that the initial number of nodes and initial total degree satisfy $D_0 = 2N_0$.

Theorem 3. *For the network evolution and influence propagation dynamics described above, the continuous-time approximation to the mean number of nodes* $n_i(t) = \mathbb{E}[N_i(t)]$ *adopting each technology is given by*

$$\begin{aligned} n_1(t) &= \alpha_1 \left(\frac{t+N_0}{N_0}\right)^{\lambda_1} + \beta_1 \left(\frac{t+N_0}{N_0}\right)^{\lambda_2} + \gamma_1, \\ n_2(t) &= \alpha_2 \left(\frac{t+N_0}{N_0}\right)^{\lambda_1} + \beta_2 \left(\frac{t+N_0}{N_0}\right)^{\lambda_2} + \gamma_2, \end{aligned} \tag{8}$$

while the mean number of indifferent nodes is

$$n_3(t) = t + N_0 - n_1(t) - n_2(t).$$

The coefficients $\alpha_i \geq 0, \beta_i, \gamma_i$ *are constants that depend on the system parameters* $q, p_{11}, p_{12}, p_{21}, p_{22}$ *and initial state* $\mathbf{X}_0$. *The exponents* λ_1 *and* λ_2 *are given by*

$$\begin{aligned} \lambda_1 &= \frac{1}{2}(1-q) + \frac{1}{4}(1+q)(p_{11} + p_{22} + \Delta), \\ \lambda_2 &= \frac{1}{2}(1-q) + \frac{1}{4}(1+q)(p_{11} + p_{22} - \Delta), \end{aligned} \tag{9}$$

where $\Delta = \sqrt{(p_{11} - p_{22})^2 + 4p_{12}p_{21}}$. *The exponents satisfy* $\lambda_2 \leq \lambda_1 \leq 1$ *and the latter equality holds if and only if*

$$p_{11} + p_{21} = p_{12} + p_{22} = 1. \tag{10}$$

Proof. The derivation of (8) and (9) is similar to the proof of Theorem 2 given in [14]. Hence, we omit the details. To establish the range of the exponents, we note that

$$\begin{aligned} \Delta &= \sqrt{(p_{11} - p_{22})^2 + 4p_{12}p_{21}} \\ &\leq \sqrt{(p_{11} - p_{22})^2 + 4(1-p_{22})(1-p_{11})} \qquad (11) \\ &= \sqrt{((p_{11} + p_{22}) - 2)^2} = 2 - p_{11} - p_{22}. \qquad (12) \end{aligned}$$

Hence, we obtain the bound $p_{11} + p_{22} + \Delta \leq 2$ and conclude that $\lambda_1 \leq 1$. Note that (11) is met with equality if and only if (10) is satisfied, i.e., $\lambda_1 = 1$ if and only if (10) holds. □

The dependence of the coefficients $\alpha_i, \beta_i, \gamma_i$ in Theorem 3 on the system parameters $\{p_{ij}\}, q$ and $\mathbf{X}_0$ is quite complex.

Several observations can be made concerning the evolution of the mean number of nodes adopting each technology and can be contrasted to the two opinion case in Section IV.

Sublinear Growth of the Market Size: We note that only the expression for the mean number of indifferent nodes $n_3(t)$ has a linear term. The mean number of nodes adopting one of the two active influences is governed by the sublinear t^{λ_1} term. According to Theorem 3, $\lambda_1 < 1$ unless nodes exposed to either form of innovation do not have the option of remaining indifferent. We omit this case from the discussion below. As a result, the fraction of each active influence within the total network population tends to zero in the long term. Nevertheless, there are two important measures to be studied: the total number of nodes adopting a new technology and the fraction of each technology among these nodes, i.e., the market size and the market share.

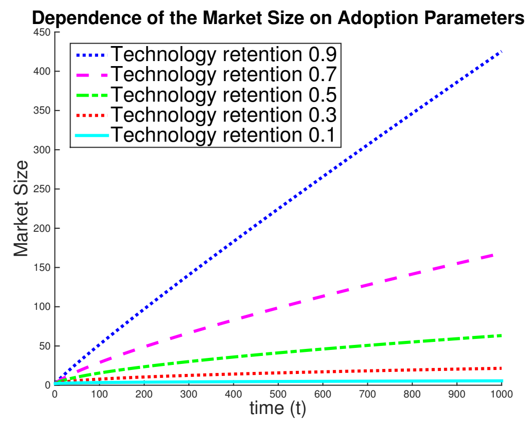

Fig. 3. The impact of the adoption parameters $p_{11} + p_{21} = p_{12} + p_{22} \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$ on the market size. The fractions of nodes that opt for the same technology as their parent node are 80% and 70%. Network evolution starts with one node of each color and $q = 0.5$.

Impact of Adoption Model on the Market Size: The size of the market is given by the total number of nodes adopting a new technology, i.e., $n_1(t) + n_2(t)$. While the market size is affected by all system parameters, the largest effect is due to the adoption parameters $\{p_{ij}\}$. In particular, the market grows monotonically with growing sums $p_{11} + p_{21}$ and $p_{12} + p_{22}$, as these sums represent the probability that a node exposed to the innovation does adopt either form of it. We call this measure the *technology retention* probability. Fig. 3 depicts the growth of the market with increasing technology retention probability.

Impact of Attachment Model on the Market Size: The growth of the market size is dominated by the $(\alpha_1 + \alpha_2)t^{\lambda_1}$ term. Hence, the largest impact of the attachment parameter q on the market size, especially in the long term, is through its effect on the exponent λ_1. Since, from (9), λ_1 is a linearly decreasing function of q for all sets of adoption parameters $\{p_{ij}\}$, the market size grows as the rate of random attachment q decreases. A higher rate of preferential attachment enables early technology adopters to establish higher prominence, thereby attracting more of the newcomer nodes to one of the technologies. Fig. 4 illustrates this effect.

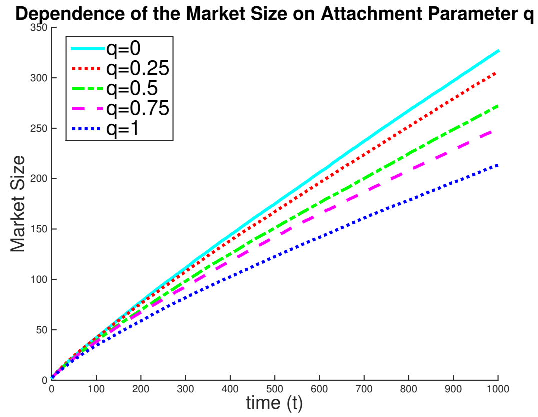

Fig. 4. The impact of the attachment parameter $q \in \{0, 0.25, 0.5, 0.75, 1\}$ on the market size. The adoption parameters are $p_{11} = 0.6, p_{21} = 0.3, p_{12} = 0.1$ and $p_{22} = 0.8$ The network evolution starts with one node of each color.

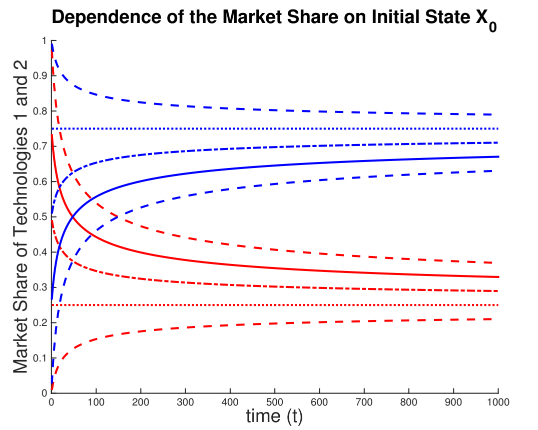

Fig. 5. The dependence of the market share on the initial network state. Fraction of initial adopters of each technology varies as $\{0, 0.25, 0.5, 0.75, 1\}$. Remaining parameters are $q = 1, p_{11} = 0.6, p_{21} = 0.3, p_{12} = 0.1$ and $p_{22} = 0.8$. Network evolution starts with one node of each color.

Long-Term Market Share Characteristics: The long-term market share of each technology is determined by the coefficients α_1 and α_2 of the t^{λ_1} term in (8). These depend not only on the adoption parameters $\{p_{ij}\}$ but also on the attachment parameter q and the initial state of the network $\mathbf{X}_0$. This dependence is in apparent contrast to the previous case of two opinions presented in Section IV, where the leading coefficients α_1 and α_2 in (7) depended only on the adoption parameters. Nevertheless, the long-term market share of each technology is not influenced by the attachment dynamics (as captured by q) nor the initial number of the early adopters (as captured by $\mathbf{X}_0$). In particular, the long-term fraction of technology 1 within the market is given by:

$$\frac{\alpha_1}{\alpha_1 + \alpha_2} = \frac{p_{11} - 2p_{12} + \Delta - p_{22}}{2(p_{11} - p_{12} + p_{21} - p_{22})}.$$

Consequently, the attachment model and the preferences of the initial adopters have only short-term effects on the market share. Fig. 5 shows the diminishing effect of the initial network composition on the evolution of the market shares with time.

These observations reiterate the suggestion that the *long-term spread of two competing influences is ultimately governed by their inherent persuasion power*. The attachment dynamics and the early adopters have only a secondary effect on the market size and market share.

VI. Conclusion

In this paper, we have introduced a new analytical model to study the battle of opinions over social networking platforms. In particular, we focused on the spread of multiple competing influences over a simultaneously evolving network. This simple yet powerful model, has allowed us to capture a range of exposure and adoption dynamics, which account for both the preferential and random nature of exposure, as well as different persuasion power of different opinions. We have analytically characterized the evolution of the mean influence spread over time as a function of the initial adoption, as well as, exposure and adoption dynamics. Our analysis, supported by two case studies for further insights, has shown that the persuasion power of an influence has the most potent effect on the extend of its spread. We have further observed how exposure dynamics determine whether the initial adopters play a short or long lived effect on the evolution of the influence spread. Our work has provided a useful new model with several potential directions for extension.

References

[1] N. A. Christakis and J. H. Fowler, "The spread of obesity in a large social network over 32 years," *New England Journal of Medicine*, vol. 357, no. 4, pp. 370–379, 2007.

[2] H. Kwak, C. Lee, H. Park, and S. Moon, "What is twitter, a social network or a news media?" in *Proc. of the 19th Int. Conf. on World Wide Web*. ACM, 2010, pp. 591–600.

[3] A.-L. Barabasi and R. Albert, "Emergence of scaling in random networks," *Science*, vol. 286, no. 5439, pp. 509–512, 1999.

[4] D. M. Pennock, G. W. Flake, S. Lawrence, E. J. Glover, and C. L. Giles, "Winners don't take all: Characterizing the competition for links on the web," *Proc. Nat. Academy of Sciences*, vol. 99, pp. 5207–5211, 2002.

[5] A. Vázquez, "Growing network with local rules: Preferential attachment, clustering hierarchy, and degree correlations," *Phys. Rev. E*, vol. 67, p. 056104, May 2003.

[6] M. O. Jackson and B. W. Rogers, "Meeting strangers and friends of friends: How random are social networks?" *American Economic Review*, vol. 97, no. 3, pp. 890–915, 2007.

[7] S. Shakkottai, M. Fomenkov, R. Koga, D. Krioukov, and K. Claffy, "Evolution of the internet as-level ecosystem," in *Complex Sciences*. Springer Berlin Heidelberg, 2009, vol. 5, pp. 1605–1616.

[8] R. Albert and A.-L. Barabási, "Statistical mechanics of complex networks," *Rev. Mod. Phys.*, vol. 74, no. 1, pp. 47–97, Jan. 2002.

[9] J. Kleinberg, R. Kumar, P. Raghavan, S. Rajagopalan, and A. Tomkins, "The web as a graph: Measurements, models, and methods," in *Computing and Combinatorics*, 1999, vol. 1627, pp. 1–17.

[10] D. Kempe, J. Kleinberg, and E. Tardos, "Maximizing the spread of influence through a social network," in *Proc. of the 9th Int. Conf. on Knowledge Discovery and Data Mining*, 2003, pp. 137–146.

[11] D. Acemoglu, A. Ozdaglar, and E. Yildiz, "Diffusion of innovations in social networks," in *Decision and Control and European Control Conf. (CDC-ECC), 2011*, Dec 2011, pp. 2329–2334.

[12] N. Berger, C. Borgs, J. T. Chayes, and A. Saberi, "On the spread of viruses on the internet," in *Proc. of the 16th Annual ACM-SIAM Symposium on Discrete Algorithms*, 2005, pp. 301–310.

[13] R. Pastor-Satorras and A. Vespignani, "Epidemic spreading in scale-free networks," *Phys. Rev. Lett.*, vol. 86, pp. 3200–3203, Apr 2001.

[14] http://newslab.ece.ohio-state.edu/research/resources/opinionbattles.pdf

[15] W. Magnus, "On the exponential solution of differential equations for a linear operator," *Communications on Pure and Applied Mathematics*, vol. 7, no. 4, pp. 649–673, 1954.