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Analysis of Precoding-Based Intersession Network Coding and The Corresponding 3-Unicast Interference Alignment Scheme

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Abstract— **Recently, a new precoding-based intersession network coding (NC) scheme has been proposed, which applies the interference alignment technique, originally devised for wireless interference channels, to the 3-unicast problem of directed acyclic networks. Motivated by the graph-theoretic characterizations of classic linear NC results, this paper investigates several key relationships between the point-to-point network channel gains and the underlying graph structure. Such relationships are critical when characterizing graph-theoretically the feasibility of precoding-based solutions. One example of the applications of our results is to answer (at least partially) the conjectures of the 3-unicast interference alignment technique and the corresponding graph-theoretic characterization conditions.**

Index Terms— **Asymptotic interference alignment, interference channels, intersession network coding, 3-unicast networks.**

I. Introduction

Characterizing the capacity or the feasibility of satisfying the network traffic demands between multiple coexisting source-destination pairs (sessions) has been a long-standing challenge. Recently, allowing the *coding operation* at the intermediate network nodes, namely *network coding*, has emerged and shown to achieve the information-theoretic capacity for the single multicast [2] even when considering only linear network codes [3]. Several papers have since studied the network code construction problem for the above single multicast setting [4]–[7].

On the other hand, when there are multiple coexisting sessions in the network, the corresponding network code design/analysis problem, also known as *intersession network coding* (INC), becomes notoriously challenging due to the potential interference within the network. For example, *linear network coding* no longer achieves the capacity [10]. Deciding the existence of a (linear) network code that satisfies general traffic demands becomes an NP-hard problem [5], [9]. Thus, the recent INC studies have focused on the optimal characterizations over some restrictive networks or limited rate constraints, including the capacity regions for *directed cycles* [14], degree-2 three-layer *directed acyclic networks* (DAG) [15], and for networks with integer link capacity and two coexisting rate-1 multicast sessions [8].

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Recently, the authors in [12], [13] applied the interference alignment (IA) technique, originally developed in wireless interference channel [11], to 3-unicast coexisting sessions in the name of 3-unicast Asymptotic Network Alignment (3-ANA). Their application of the interference-aligning idea brings a new perspective to the INC problems, which enables us to focus only on designing the precoding and decoding mappings at the sources and destinations while allowing randomly generated *local encoding kernels* [7] within the network. Compared to the classic algebraic framework that fully controls the encoder, decoder, and *local encoding kernels* [5], this *precoding*-based framework can tradeoff the ultimate achievable throughput with a distributed, implementation-friendly structure that allows pure random linear NC in the interior of the network. Their initial study on 3-unicast networks shows that, by performing precoding across multiple time slots and applying the IA technique, the *precoding*-based NC can perform strictly better than the pure routing in some networks, and even better than any linear NC solutions in some networks. Such results thus demonstrate a new balance between practicality and throughput enhancement. Further development of the *precoding*-based framework could thus have significant impact on the practical NC design.

In this work, we first study several basic properties of the *precoding*-based framework, and then apply our results to the 3-ANA scheme proposed in [12], [13], which derives the certain algebraic but computationally intractable conditions for the 3-ANA scheme to achieve asymptotically half of the interference-free throughput for each transmission pair. Note that in the wireless interference channels where the IA technique was originally developed, those algebraic feasibility conditions can be satisfied with high probability provided the channel gains are independent and continuously distributed random variables [11]. For comparison, the "network channel gains" are generally correlated and the correlation depends heavily on the underlying network topology [12], [13]. As a result, we need new efficient ways to decide whether the network of interest admits a 3-ANA scheme that achieves half of the interference-free throughput asymptotically as promised in the wireless interference channels. The results in this work answer this question by developing new graph-theoretic conditions which are equivalent (at least partially) to the feasibility of the 3-ANA scheme. The proposed graph-theoretic conditions can be easily computed and checked within polynomial time.

II. Precoding-Based Intersession NC

A. System Model and Some Graph-Theoretic Definitions

Consider a DAG $G=(V,E)$ where V is the set of nodes and E is the set of directed edges. Each edge $e\in E$ is represented by $e=uv$, where $u=\mathsf{tail}(e)$ and $v=\mathsf{head}(e)$ are the tail and head of e, respectively. For any node $v\in V$, use $\mathsf{In}(v)\subseteq E$ to denote the collection of incoming edges $uv\in E$. Similarly, $\mathsf{Out}(v)\subseteq E$ contains all $vw\in E$.

A path P is a series of adjacent edges $e_1e_2\cdots e_k$ where $\mathsf{head}(e_i)=\mathsf{tail}(e_{i+1})\,\forall\, i\in\{1,\cdots,k-1\}$. We say that e_1 and e_k are the starting and ending edges of P, respectively. For any path P, we use $e\in P$ to indicate that an edge e is used by P. For a given path Q, xQy denotes the path segment of Q from node x (edge x) to node y (edge y). A path starting from edge e_1 and ending at edge e_k is denoted by $P_{e_1e_k}$. When $u=\mathsf{tail}(e_1)$ and $v=\mathsf{head}(e_k)$, we sometimes use P_{uv} to emphasize the starting and ending nodes of $P_{e_1e_k}$. We say a node u is an *upstream* node of a node v (or v is a *downstream* node of u) if there exists a path P_{uv} and denote it as $u\prec v$. If neither $u\prec v$ nor $u\succ v$, then we say that u and v are *not reachable* with each other. Edge e_1 is an *upstream* edge of e_2 if $\mathsf{head}(e_1)\preceq\mathsf{tail}(e_2)$, and denoted as $e_1\prec e_2$. Similarly, e_1 and e_2 are *not reachable* with each other, if neither $e_1\prec e_2$ nor $e_1\succ e_2$. A *k-edge cut* (sometimes just the "edge cut") separating node sets $U\subseteq V$ and $W\subseteq V$ is a collection of k edges such that any path from $u\in U$ to $w\in W$ must use at least one of those k edges. The value of an edge cut is the number of edges in the cut (A *k-edge cut* has value k). We denote the minimum value among all the *edge cuts* separating U and W as $\mathsf{EC}(U;W)$. Then, $\mathsf{EC}(U;W)=0$ when they are already disconnected. By convention, if $U\cap W\neq\emptyset$, we define $\mathsf{EC}(U;W)=\infty$. We also denote the collection of all distinct 1-*edge cuts* separating U and W as $\mathsf{1cut}(U;W)$. Please refer to [17] for more detailed graph-theoretic definitions.

B. Algebraic Framework

Given a DAG $G=(V,E)$, we consider the multiple-unicast problem. For a source-destination pair (s_k,d_k), let l_k be the number of information symbols to be delivered. Each information symbol is chosen from a finite field $\mathbb{F}_q$ with some sufficiently large q. We use $\mathbb{F}$ as shorthand of $\mathbb{F}_q$.

Following the widely-used instant-transmission model on a DAG [5], we assume that each edge is capable of transmitting one symbol in $\mathbb{F}$ for any given time without delay and consider *linear network coding* over the entire network, i.e., a symbol on an edge $e\in E$ is a $\mathbb{F}$-linear combination of the symbols on its adjacent incoming edges of $\mathsf{In}(\mathsf{tail}(e))$. The collection of coefficients (i.e. *local encoding kernels*) used for such linear combinations are represented by the network variables $\underline{\mathbf{x}}$, defined as the collection of the variable $x_{e'e''}$ for all adjacent edge pairs (e',e'') (i.e., $\mathsf{head}(e')=\mathsf{tail}(e'')$). See [5] for the detailed discussion. Following this notation, the channel gain $m_{e_1;e_2}(\underline{\mathbf{x}})$ from an edge e_1 to an edge e_2 can be written as a polynomial in the ring of polynomials with respect to $\underline{\mathbf{x}}$ over $\mathbb{F}$. More rigorously, it can be defined as

$$m_{e_1;e_2}(\underline{\mathbf{x}})=\sum_{\forall P_{e_1e_2}\in\mathbf{P}_{e_1e_2}}\left(\prod_{\forall\, e',\,e''\in P_{e_1e_2}\text{ where }\mathsf{head}(e')=\mathsf{tail}(e'')} x_{e'e''}\right)$$

where $\mathbf{P}_{e_1e_2}$ denotes the collection of all distinct $P_{e_1e_2}$ paths.

Only when $e_1\preceq e_2$, $m_{e_1;e_2}(\underline{\mathbf{x}})$ is a non-zero polynomial, and we set $m_{e_1;e_2}(\underline{\mathbf{x}})=1$ when $e_1=e_2$ by convention [5]. A channel gain from a node u to a node v is defined by an $|\mathsf{In}(v)|\times|\mathsf{Out}(u)|$ polynomial matrix $\mathbf{M}_{u;v}(\underline{\mathbf{x}})$, where its (i,j)-th entry is the channel gain from j-th outgoing edge of u to i-th incoming edge of v. When considering source s_i and destination d_j, we use $\mathbf{M}_{i;j}(\underline{\mathbf{x}})$ as shorthand for $\mathbf{M}_{s_i;d_j}(\underline{\mathbf{x}})$.

We allow the *precoding*-based NC to code across τ number of time slots, which are termed as the precoding frame. The network variables corresponding to each time slot t is denoted as $\underline{\mathbf{x}}^{(t)}$, and the corresponding channel gain from s_i to d_j becomes $\mathbf{M}_{i;j}(\underline{\mathbf{x}}^{(t)})$ for all $t=1,\cdots,\tau$. Let $\mathbf{z}_i\in\mathbb{F}^{l_i\times 1}$ be the set of to-be-sent information symbols from s_i. Then, for every time slot $t=1,\cdots,\tau$, if we define the precoding matrix $\mathbf{V}_i^{(t)}\in\mathbb{F}^{|\mathsf{Out}(s_i)|\times l_i}$ for each source s_i, then each d_j receives a $|\mathsf{In}(d_j)|$-dimensional column vector $\mathbf{y}_j^{(t)}(\underline{\mathbf{x}}^{(t)})$ as

$$\mathbf{y}_j^{(t)}(\underline{\mathbf{x}}^{(t)})=\mathbf{M}_{j;j}(\underline{\mathbf{x}}^{(t)})\mathbf{V}_j^{(t)}\mathbf{z}_j+\sum_{\forall i\text{ s.t. }i\neq j}\mathbf{M}_{i;j}(\underline{\mathbf{x}}^{(t)})\mathbf{V}_i^{(t)}\mathbf{z}_i.$$

This system model can be equivalently expressed as

$$\overline{\mathbf{y}}_j=\overline{\mathbf{M}}_{j;j}\overline{\mathbf{V}}_j\,\mathbf{z}_j+\sum_{\forall i\text{ s.t. }i\neq j}\overline{\mathbf{M}}_{i;j}\overline{\mathbf{V}}_i\,\mathbf{z}_i \qquad (1)$$

where $\overline{\mathbf{V}}_i$ is the overall precoding matrix for each source s_i by vertically concatenating $\{\mathbf{V}_i^{(t)}\}_{t=1}^{\tau}$, and $\overline{\mathbf{y}}_j$ is the vertical concatenation of $\{\mathbf{y}_j^{(t)}(\underline{\mathbf{x}}^{(t)})\}_{t=1}^{\tau}$. The overall channel matrix $\overline{\mathbf{M}}_{i;j}$ is a block-diagonal polynomial matrix with $\{\mathbf{M}_{i;j}(\underline{\mathbf{x}}^{(t)})\}_{t=1}^{\tau}$ as its diagonal blocks, thus dependent on the set of network variables $\{\underline{\mathbf{x}}^{(t)}\}_{t=1}^{\tau}$.

After receiving packets for τ time slots, each destination d_j applies the overall decoding matrix $\overline{\mathbf{U}}_j\in\mathbb{F}^{l_j\times\tau\cdot|\mathsf{In}(d_j)|}$. Then, the decoded message vector $\hat{\mathbf{z}}_j$ can be expressed as

$$\hat{\mathbf{z}}_j=\overline{\mathbf{U}}_j\overline{\mathbf{y}}_j=\overline{\mathbf{U}}_j\overline{\mathbf{M}}_{j;j}\overline{\mathbf{V}}_j\,\mathbf{z}_j+\sum_{\forall i\text{ s.t. }i\neq j}\overline{\mathbf{U}}_j\overline{\mathbf{M}}_{i;j}\overline{\mathbf{V}}_i\,\mathbf{z}_i. \qquad (2)$$

We say the *precoding*-based NC problem is feasible if there exists a pair of encoding and decoding matrices $\{\overline{\mathbf{V}}_i,\forall\, i\}$ and $\{\overline{\mathbf{U}}_j,\forall\, j\}$ (which may be a function of $\{\underline{\mathbf{x}}^{(t)}\}_{t=1}^{\tau}$) such that when choosing each element of the set of network variables $\{\underline{\mathbf{x}}^{(t)}\}_{t=1}^{\tau}$ independently and uniformly randomly from $\mathbb{F}$, they satisfies

$$\begin{aligned}\overline{\mathbf{U}}_j\overline{\mathbf{M}}_{i;j}\overline{\mathbf{V}}_i&=\mathbf{I}\,(\text{identity})\quad\forall\, i=j\\ \overline{\mathbf{U}}_j\overline{\mathbf{M}}_{i;j}\overline{\mathbf{V}}_i&=\mathbf{0}\quad\forall\, i\neq j.\end{aligned} \qquad (3)$$

Remark 1: One can easily check by the cut-set bound that a necessary condition for the feasibility of a *precoding*-based NC problem is given by the frame size $\tau\geq\max_k\{l_k/\mathsf{EC}(s_k;d_k)\}$.

Remark 2: Depending on the time relationship of $\overline{\mathbf{V}}_i$ and $\overline{\mathbf{U}}_j$ with respect to the network variables $\{\underline{\mathbf{x}}^{(t)}\}_{t=1}^{\tau}$, the *precoding*-based NC can be classified as causal vs. non-causal and time-varying vs. time-invariant schemes.

C. Comparison to the Classic NC Framework

The authors in [5] established the algebraic framework for the *linear network coding* problem, which has the encoding and decoding equations very similar to (1) and (2) with the same algebraic feasibility conditions of (3).[1] The main difference between the *precoding*-based framework and the classic framework is that the latter allows the NC designer to control the network variables $\underline{\mathbf{x}}$ while the former assumes the entries of $\underline{\mathbf{x}}$ are chosen independently and uniformly. One can thus view the *precoding*-based NC as a distributed version of classic NC that tradeoffs the ultimate achievable performance for more practical distributed implementation (not controlling the behavior in the interior of the network).

One challenge when using the algebraic feasibility conditions (3) is that given a network code, it is easy to verify whether (3) is satisfied or not, but it is notoriously hard to decide the existence of a NC solution satisfying (3) [5], [9]. Only in some special scenarios, we can convert those algebraic conditions into some graph-theoretic conditions for which one can decide the existence of a feasible network code in polynomial time. For example, if there exists only a single session in the network (s_1, d_1), then the existence of a NC solution satisfying (3) is equivalent to the time-averaged rate l_1/τ being no larger than $\mathsf{EC}(s_1; d_1)$. Moreover, if $l_1/\tau \leq \mathsf{EC}(s_1; d_1)$, then we can use random linear network coding [7] to construct the optimal network code. Another example is when there are only two sessions (s_1, d_1) and (s_2, d_2) with $l_1 = l_2 = \tau = 1$. Then, the existence of a network code satisfying (3) is equivalent to the conditions that the 1-*edge cuts* in the network who perform the interference-cancelling are properly placed in certain ways [8]. Based on the above observation, the main focus of this work is to develop new graph-theoretic conditions for a special scenario of the *precoding*-based NC, the 3-unicast ANA scheme.

D. Special Example of Precoding-Based Framework: 3-ANA Scheme

Before proceeding, we introduce some algebraic definitions. We say that a set of polynomials $\mathbf{h}(\underline{\mathbf{x}}) = \{h_1(\underline{\mathbf{x}}), ..., h_N(\underline{\mathbf{x}})\}$ is linearly dependent if and only if $\sum_{k=1}^{N} \alpha_k h_k(\underline{\mathbf{x}}) = 0$ for some coefficients $\{\alpha_k\}_{k=1}^{N}$ that are not all zeros. By treating $\mathbf{h}(\underline{\mathbf{x}}^{(k)})$ as a polynomial row vector and vertically concatenating them together, we have an $M\times N$ polynomial matrix $[\mathbf{h}(\underline{\mathbf{x}}^{(k)})]_{k=1}^{M}$. Namely, each row is based on the same set of polynomials $\mathbf{h}(\underline{\mathbf{x}})$ but with different variables $\underline{\mathbf{x}}^{(k)}$ for each row k, respectively. We say that the polynomial matrix $[\mathbf{h}(\underline{\mathbf{x}}^{(k)})]_{k=1}^{M}$ is generated from $\mathbf{h}(\underline{\mathbf{x}})$. When considering only two polynomials $g(\underline{\mathbf{x}})$ and $h(\underline{\mathbf{x}})$, we say $g(\underline{\mathbf{x}})$ and $h(\underline{\mathbf{x}})$ are *equivalent*, denoted by $g(\underline{\mathbf{x}}) \equiv h(\underline{\mathbf{x}})$, if $\{g(\underline{\mathbf{x}}), h(\underline{\mathbf{x}})\}$ is linearly dependent. Similarly, $g(\underline{\mathbf{x}})$ and $h(\underline{\mathbf{x}})$ are *not equivalent*, denoted by $g(\underline{\mathbf{x}}) \not\equiv h(\underline{\mathbf{x}})$, if $\{g(\underline{\mathbf{x}}), h(\underline{\mathbf{x}})\}$ is linearly independent. We use $\mathsf{GCD}(\, g(\underline{\mathbf{x}}), h(\underline{\mathbf{x}}))$ to denote the greatest common factor of the two polynomials.

We now consider a special class of networks, called the 3-ANA network: A network G is a 3-ANA network if (i) there are 3 source-destination pairs, $\{(s_i, d_i)\}_{i=1}^{3}$, where all sources and destinations are distinct; (ii) the topology of G is stable over a precoding frame; (iii) $|\mathsf{In}(s_i)| = 0$ and $|\mathsf{Out}(s_i)| = 1\ \forall i$ (let the only outgoing edge of s_i as e_{s_i}.); (iv) $|\mathsf{In}(d_j)| = 1$ and $|\mathsf{Out}(d_j)| = 0\ \forall j$ (let the only incoming edge of d_j as e_{d_j}.); and (v) any d_j can be reached from any s_i. (otherwise, it becomes trivial [12].) Note that $\mathbf{M}_{i;j}(\underline{\mathbf{x}})$ becomes a single quantity by (iii) and (iv) (from e_{s_i} to e_{d_j}), thus we denote it as $m_{ij}(\underline{\mathbf{x}})$ for shorthand. We use $G_{3\text{ANA}}$ to emphasize that we are focusing on this 3-ANA network.

The authors in [12], [13] applied the interference alignment technique to construct the precoding matrices $\{\overline{\mathbf{V}}_i, \forall i\}$ for the above 3-ANA network. Namely, consider the following parameter values: $\tau = 2n+1$, $l_1 = n+1$, $l_2 = n$, and $l_3 = n$ for some positive integer n. The goal is thus to achieve the rate tuple $(\frac{n+1}{2n+1}, \frac{n}{2n+1}, \frac{n}{2n+1})$ in a 3-ANA network by applying the following $\{\overline{\mathbf{V}}_i, \forall i\}$ construction methodology of [11]: Define $L(\underline{\mathbf{x}}) = m_{13}(\underline{\mathbf{x}}) m_{32}(\underline{\mathbf{x}}) m_{21}(\underline{\mathbf{x}})$ and $R(\underline{\mathbf{x}}) = m_{12}(\underline{\mathbf{x}}) m_{23}(\underline{\mathbf{x}}) m_{31}(\underline{\mathbf{x}})$, and consider the following 3 row vectors of dimensions $n+1$, n, and n, respectively (Each entry of these row vectors is a polynomial with respect to $\underline{\mathbf{x}}$ but we drop the input argument $\underline{\mathbf{x}}$ for simplicity.):

$$\mathbf{v}_1^{(n)}(\underline{\mathbf{x}}) = m_{23} m_{32} \left[R^n, \ \ R^{n-1}L, \ \ ..., \ \ RL^{n-1}, \ \ L^n \right],$$
$$\mathbf{v}_2^{(n)}(\underline{\mathbf{x}}) = m_{13} m_{32} \left[R^n, \ \ R^{n-1}L, \ \ ..., \ \ RL^{n-1} \right], \text{ and}$$
$$\mathbf{v}_3^{(n)}(\underline{\mathbf{x}}) = m_{12} m_{23} \left[R^{n-1}L, \ \ ..., \ \ RL^{n-1}, \ \ L^n \right],$$

where the superscript (n) is to emphasize the current n value used in the construction. The precoding matrix for each time slot t is thus constructed as $\mathbf{V}_i^{(t)} = \mathbf{v}_i^{(n)}(\underline{\mathbf{x}}^{(t)})$, so that their vertical concatenation, i.e., the overall precoding matrix becomes (recall $\tau = 2n+1$) $\overline{\mathbf{V}}_i = [\mathbf{v}_i^{(n)}(\underline{\mathbf{x}}^{(t)})]_{t=1}^{2n+1}$. Then, the above construction achieves the desired rates $(\frac{n+1}{2n+1}, \frac{n}{2n+1}, \frac{n}{2n+1})$ if the overall precoding matrices $\overline{\mathbf{V}}_i$ satisfy the following six constraints:

$$d_1 : \langle\, \overline{\mathbf{M}}_{3;1}\overline{\mathbf{V}}_3 \,\rangle = \langle\, \overline{\mathbf{M}}_{2;1}\overline{\mathbf{V}}_2 \,\rangle \tag{4}$$
$$\mathsf{rank}(\mathbf{S}_1^{(n)}) = \mathsf{rank}(\left[\, \overline{\mathbf{M}}_{1;1}\overline{\mathbf{V}}_1 \ \ \overline{\mathbf{M}}_{2;1}\overline{\mathbf{V}}_2 \,\right]) = 2n+1 \tag{5}$$
$$d_2 : \langle\, \overline{\mathbf{M}}_{3;2}\overline{\mathbf{V}}_3 \,\rangle \subseteq \langle\, \overline{\mathbf{M}}_{1;2}\overline{\mathbf{V}}_1 \,\rangle \tag{6}$$
$$\mathsf{rank}(\mathbf{S}_2^{(n)}) = \mathsf{rank}(\left[\, \overline{\mathbf{M}}_{2;2}\overline{\mathbf{V}}_2 \ \ \overline{\mathbf{M}}_{1;2}\overline{\mathbf{V}}_1 \,\right]) = 2n+1 \tag{7}$$
$$d_3 : \langle\, \overline{\mathbf{M}}_{2;3}\overline{\mathbf{V}}_2 \,\rangle \subseteq \langle\, \overline{\mathbf{M}}_{1;3}\overline{\mathbf{V}}_1 \,\rangle \tag{8}$$
$$\mathsf{rank}(\mathbf{S}_3^{(n)}) = \mathsf{rank}(\left[\, \overline{\mathbf{M}}_{3;3}\overline{\mathbf{V}}_3 \ \ \overline{\mathbf{M}}_{1;3}\overline{\mathbf{V}}_1 \,\right]) = 2n+1 \tag{9}$$

where $\langle\, \mathbf{A} \,\rangle$ and $\mathsf{rank}(\mathbf{A})$ denote the column vector space and the rank, respectively, of a given matrix $\mathbf{A}$. Recalling the definition of $\overline{\mathbf{M}}_{i;j}$ in Section III-B, the choice of $\tau = 2n+1$ and the assumption of $|\mathsf{Out}(s_i)| = |\mathsf{In}(d_j)| = 1$ result in $\overline{\mathbf{M}}_{i;j}$ being a $(2n+1)\times(2n+1)$ diagonal matrix with the t-th diagonal element $m_{ij}(\underline{\mathbf{x}}^{(t)})$. Thus from the stable network assumption of (ii) and the constructed $\{\overline{\mathbf{V}}_i, \forall i\}$, the square matrices $\{\mathbf{S}_i^{(n)}, \forall i\}$ become all *row-invarint*.

[1]The original work [5] focuses on the single time slot $\tau = 1$, although the results can be easily generalized for $\tau > 1$ as well. Note that $\tau > 1$ provides a greater degree of freedom when designing the coding matrices $\{\overline{\mathbf{V}}_i, \forall i\}$ and $\{\overline{\mathbf{U}}_j, \forall j\}$. Such *time extension* turns out to be especially critical in a *precoding*-based NC design as it is generally much harder to design $\{\overline{\mathbf{V}}_i, \forall i\}$ and $\{\overline{\mathbf{U}}_j, \forall j\}$ for randomly chosen $\underline{\mathbf{x}}$ when $\tau = 1$. An example of this time extension will be discussed in Section III-D.

The interpretation of the above constraints is straightforward. In order for the interference at d_1 to be aligned, the precoding matrices $\{\overline{\mathbf{V}}_i\}_{i=1}^3$ must be designed such that (4) can be satisfied. Note that by simple linear algebra, $\mathsf{rank}(\overline{\mathbf{M}}_{2;1}\overline{\mathbf{V}}_2) = \mathsf{rank}(\overline{\mathbf{M}}_{3;1}\overline{\mathbf{V}}_3) \leq n$ and $\mathsf{rank}(\overline{\mathbf{M}}_{1;1}\overline{\mathbf{V}}_1) \leq n+1$. (5) thus guarantees that $\mathsf{rank}([\,\overline{\mathbf{M}}_{1;1}\overline{\mathbf{V}}_1 \;\; \overline{\mathbf{M}}_{2;1}\overline{\mathbf{V}}_2\,]) = \mathsf{rank}(\overline{\mathbf{M}}_{1;1}\overline{\mathbf{V}}_1) + \mathsf{rank}(\overline{\mathbf{M}}_{2;1}\overline{\mathbf{V}}_2)$ and $\mathsf{rank}(\overline{\mathbf{M}}_{1;1}\overline{\mathbf{V}}_1) = n+1$, implying that d_1 can successfully remove the aligned interference while recovering all $l_1 = n+1$ information symbols intended for d_1. Similar arguments can be used to justify (6) to (9) from the perspectives of d_2 and d_3.

By noticing the special Vandermonde form when constructing $\overline{\mathbf{V}}_i$, it is trivial that (4), (6), and (8) hold simultaneously on the $G_{3\text{ANA}}$ of interest when it satisfies $L(\underline{\mathbf{x}}) \not\equiv R(\underline{\mathbf{x}})$. The authors in [13] further derived that when $L(\underline{\mathbf{x}}) \not\equiv R(\underline{\mathbf{x}})$, the conditions (5), (7), and (9) hold with high probability if the following algebraic conditions are satisfied:

$$m_{11}m_{23}\sum_{i=0}^{n}\alpha_i\big(L/R\big)^i \neq m_{13}m_{21}\sum_{j=0}^{n-1}\beta_j\big(L/R\big)^j \quad (10)$$

$$m_{22}m_{13}\sum_{i=0}^{n}\alpha_i\big(L/R\big)^i \neq m_{23}m_{12}\sum_{j=0}^{n-1}\beta_j\big(L/R\big)^j \quad (11)$$

$$m_{33}m_{12}\sum_{i=0}^{n}\alpha_i\big(L/R\big)^i \neq m_{32}m_{13}\sum_{j=0}^{n-1}\beta_j\big(L/R\big)^j \quad (12)$$

i.e., if the $G_{3\text{ANA}}$ of interest satisfies (10), (11), and (12) $\forall\,\alpha_i, \beta_j \in \mathbb{F}$ except all-zeros, then they are sufficient for (5), (7), and (9), respectively, w.h.p. when $L(\underline{\mathbf{x}}) \not\equiv R(\underline{\mathbf{x}})$.

It can be easily shown that directly verifying the above sufficient conditions is computationally intractable. The following conjecture is thus proposed in [13] to reduce their computational complexity.

Conjecture (Page 3, [13]): For any n value used in the 3-ANA construction,

$$m_{11}m_{23} \not\equiv m_{13}m_{21} \quad \text{and} \quad m_{11}m_{32} \not\equiv m_{12}m_{31}, \quad (13)$$

$$m_{22}m_{13} \not\equiv m_{23}m_{12} \quad \text{and} \quad m_{22}m_{31} \not\equiv m_{21}m_{32}, \quad (14)$$

$$m_{33}m_{12} \not\equiv m_{32}m_{13} \quad \text{and} \quad m_{33}m_{21} \not\equiv m_{31}m_{23}, \quad (15)$$

(13), (14), and (15) are sufficient for the $G_{3\text{ANA}}$ of interest to satisfy (10), (11), and (12) $\forall\,\alpha_i, \beta_j \in \mathbb{F}$ except all-zeros, respectively, when $L(\underline{\mathbf{x}}) \not\equiv R(\underline{\mathbf{x}})$ holds.

Whether the above conjecture is indeed true or not remains an open problem. (Currently, all numerical experiments support this conjecture [13].) Note that even if the conjecture is true, we still need to check $L(\underline{\mathbf{x}}) \not\equiv R(\underline{\mathbf{x}})$, which is highly non-trivial for large networks. One main result of this work (Propositions 1 and 2) is to identify some graph-theoretic condition which can be easily verified in polynomial time, with the algebraic condition of $L(\underline{\mathbf{x}}) \not\equiv R(\underline{\mathbf{x}})$. The second main result (Proposition 2) is to prove the conjecture positively for the simplest case of $n = 1$.

Remark: In the setting of wireless interference channels, the individual channel gains are independently and continuously distributed, for which one can easily prove that $L(\underline{\mathbf{x}}) \not\equiv R(\underline{\mathbf{x}})$ with close-to-one probability. For a network setting, the channel gains $\mathbf{M}_{i;j}(\underline{\mathbf{x}})$ are no longer independent

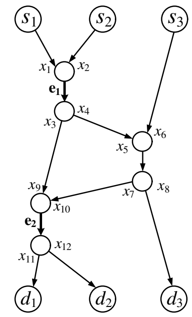

Fig. 1. Example $G_{3\text{ANA}}$ structure satisfying $L(\underline{\mathbf{x}}) \equiv R(\underline{\mathbf{x}})$ with $\underline{\mathbf{x}} = \{x_1, x_2, ..., x_{12}\}$.

for different (i, j) pairs and the correlation depends on the underlying network topology, such as the 3-ANA network example satisfying $L(\underline{\mathbf{x}}) \equiv R(\underline{\mathbf{x}})$ described in Fig. 1.

III. Properties of Precoding-Based Framework

In this section, we provide a few fundamental relationships between the channel gains and underlying DAG G in the *precoding*-based framework. These newly discovered results will later be used to prove the graph-theoretic condition of the 3-ANA scheme. For the ease of exposition, we begin by simplifying the feasibility conditions of 3-ANA about the *row-invariant* matrices $\{\mathbf{S}_i^{(n)}\,\forall\, i\}$.

A. From Non-Zero Determinant to Linear Independence

Theorem 1: Fix any integer $N > 1$. Consider a set of N non-zero polynomials $\mathbf{h}(\underline{\mathbf{x}}) = \{h_1(\underline{\mathbf{x}}), ..., h_N(\underline{\mathbf{x}})\}$ and the polynomial matrix $[\mathbf{h}(\underline{\mathbf{x}}^{(k)})]_{k=1}^N$ generated from $\mathbf{h}(\underline{\mathbf{x}})$. Then, assuming sufficiently large finite field size q, $\mathbf{h}(\underline{\mathbf{x}})$ is linearly dependent *if and only if* $\mathsf{det}([\mathbf{h}(\underline{\mathbf{x}}^{(k)})]_{k=1}^N) = 0$.

Proof of $\Rightarrow$: Suppose that $\mathbf{h}(\underline{\mathbf{x}})$ is linearly dependent. Then, there exists a set of coefficients $\{\alpha_k\}_{k=1}^N$ such that $\sum_{k=1}^N \alpha_k h_k(\underline{\mathbf{x}}) = 0$ and at least one of them is non-zero. Since $[\mathbf{h}(\underline{\mathbf{x}}^{(k)})]_{k=1}^N$ is *row-invariant*, we can perform elementary column operations on $[\mathbf{h}(\underline{\mathbf{x}}^{(k)})]_{k=1}^N$ using $\{\alpha_k\}_{k=1}^N$ to create an all-zero column. Thus, $\mathsf{det}([\mathbf{h}(\underline{\mathbf{x}}^{(k)})]_{k=1}^N) = 0$.

Proof of $\Leftarrow$: We prove this direction by induction on the value of N. When $N = 2$, $\mathsf{det}([\mathbf{h}(\underline{\mathbf{x}}^{(k)})]_{k=1}^N) = 0$ implies $h_1(\underline{\mathbf{x}}^{(1)})h_2(\underline{\mathbf{x}}^{(2)}) = h_2(\underline{\mathbf{x}}^{(1)})h_1(\underline{\mathbf{x}}^{(2)})$, thereby it must be $h_1(\underline{\mathbf{x}}) = h_2(\underline{\mathbf{x}})$ such that $\mathbf{h}(\underline{\mathbf{x}}) = \{h_1(\underline{\mathbf{x}}), h_2(\underline{\mathbf{x}})\}$ is linearly dependent. Suppose that this holds for any $N < n_0$. When $N = n_0$, consider the (1,1)-th cofactor $C_{11}(\underline{\mathbf{x}}^{(2)}, ..., \underline{\mathbf{x}}^{(n_0)})$ by removing 1st row and 1st column from $[\mathbf{h}(\underline{\mathbf{x}}^{(k)})]_{k=1}^{n_0}$.

Consider the following two cases. When C_{11} is a zero polynomial, the induction tells us that $\{h_2(\underline{\mathbf{x}}), ..., h_{n_0}(\underline{\mathbf{x}})\}$ is linearly dependent, thereby so is $\mathbf{h}(\underline{\mathbf{x}})$ by definition. When C_{11} is a non-zero polynomial, since we assume the sufficiently large enough q, there exists an assignment from $\hat{\mathbf{x}}_2$ to $\hat{\mathbf{x}}_{n_0}$ such that $C_{11}(\hat{\mathbf{x}}_2, ..., \hat{\mathbf{x}}_{n_0}) \neq 0$. But note that by the Laplace expansion, we also have $\sum_{k=1}^{n_0} h_k(\underline{\mathbf{x}}^{(1)})C_{1k} = 0$ where C_{1k} is the $(1, k)$-th cofactor. By evaluating C_{1k} with $\{\hat{\mathbf{x}}_i\}_{i=2}^{n_0}$, we can conclude that $\mathbf{h}(\underline{\mathbf{x}})$ is linearly dependent since $C_{11}(\hat{\mathbf{x}}_2, ..., \hat{\mathbf{x}}_{n_0}) \neq 0$. ■

Remark: Theorem 1 can be rewritten such that when $N > 1$, a set of N non-zero polynomials $\mathbf{h}(\underline{\mathbf{x}})$ is linearly independent *if and only if* there exists some assignments $\{\hat{\mathbf{x}}_k\}_{k=1}^N$ resulting $\mathsf{det}([\mathbf{h}(\hat{\mathbf{x}}_k)]_{k=1}^N) \neq 0$. By the assumption of sufficiently large field size q and Schwartz-Zippel lemma, it is also equivalent to that $\mathsf{det}([\mathbf{h}(\underline{\mathbf{x}}^{(k)})]_{k=1}^N) \neq 0$ w.h.p.

Theorem 1 is important in a sense that this enables us to simplify the feasibility characterization of 3-ANA. From the construction in Section II-D, the *row-invariant* matrix $\mathbf{S}_i^{(n)}$ is in the form of $\mathbf{S}_i^{(n)} = [\mathbf{h}_i^{(n)}(\underline{\mathbf{x}}^{(t)})]_{t=1}^{(2n+1)}$ where $\mathbf{h}_1^{(n)}(\underline{\mathbf{x}}) = [m_{11}(\underline{\mathbf{x}})\mathbf{v}_1^{(n)}(\underline{\mathbf{x}}),\ m_{21}(\underline{\mathbf{x}})\mathbf{v}_2^{(n)}(\underline{\mathbf{x}})]$, $\mathbf{h}_2^{(n)}(\underline{\mathbf{x}}) = [m_{22}(\underline{\mathbf{x}})\mathbf{v}_2^{(n)}(\underline{\mathbf{x}}),\ m_{12}(\underline{\mathbf{x}})\mathbf{v}_1^{(n)}(\underline{\mathbf{x}})]$, and $\mathbf{h}_3^{(n)}(\underline{\mathbf{x}}) = [m_{33}(\underline{\mathbf{x}})\mathbf{v}_3^{(n)}(\underline{\mathbf{x}}),\ m_{13}\mathbf{v}_1^{(n)}(\underline{\mathbf{x}})]$. By the assumptions of 3-ANA network, the polynomials in the set $\mathbf{h}_i^{(n)}(\underline{\mathbf{x}})$ are all non-zero. Then, the linear independence of $\mathbf{h}_1^{(n)}(\underline{\mathbf{x}})$ can be used to prove that (10) is not only sufficient but also necessary for (5) w.h.p. when $L(\underline{\mathbf{x}}) \not\equiv R(\underline{\mathbf{x}})$. By similar arguments on $\mathbf{h}_2^{(n)}(\underline{\mathbf{x}})$ and $\mathbf{h}_3^{(n)}(\underline{\mathbf{x}})$, (11) and (12) are both necessary and sufficient for (7) and (9), respectively, w.h.p. when $L(\underline{\mathbf{x}}) \not\equiv R(\underline{\mathbf{x}})$.

B. Subgraph Property of Precoding-Based Framework

Given a DAG G, recall the definition of channel gain $m_{e_1;e_2}(\underline{\mathbf{x}})$ from e_1 to e_2 in Section II-B. For a subgraph $G' \subseteq G$ containing e_1 and e_2, let $m_{e_1;e_2}(\underline{\mathbf{x}}')$ denote the channel gain from e_1 to e_2 in G'.

Theorem 2 (Subgraph Property): Given a DAG G, consider an arbitrary, but fixed, finite collection of edge pairs, $\{(e_i, e_i') \in E^2 : e_i \preceq e_i' \text{ and } i \in I\}$ where I is a finite index set, and two non-zero polynomial functions $f : \mathbb{F}^{|I|} \mapsto \mathbb{F}$ and $g : \mathbb{F}^{|I|} \mapsto \mathbb{F}$. Then, $f(\{m_{e_i;e_i'}(\underline{\mathbf{x}})\,\forall i \in I\}) \equiv g(\{m_{e_i;e_i'}(\underline{\mathbf{x}})\,\forall i \in I\})$ *if and only if* for all subgraphs $G' \subseteq G$ containing all edges in $\{e_i, e_i'\,\forall i \in I\}$, $f(\{m_{e_i;e_i'}(\underline{\mathbf{x}}')\,\forall i \in I\}) \equiv g(\{m_{e_i;e_i'}(\underline{\mathbf{x}}')\,\forall i \in I\})$.

Proof of $\Leftarrow$: Choosing $G' = G$ simply proves.

Proof of $\Rightarrow$: Since $f(\{m_{e_i;e_i'}(\underline{\mathbf{x}})\,\forall i \in I\}) \equiv g(\{m_{e_i;e_i'}(\underline{\mathbf{x}})\,\forall i \in I\})$, we can assume $f(\{m_{e_i;e_i'}(\underline{\mathbf{x}})\,\forall i \in I\}) = \alpha g(\{m_{e_i;e_i'}(\underline{\mathbf{x}})\,\forall i \in I\})$ for some $\alpha \in \mathbb{F}\backslash\{0\}$. Consider a subgraph G' containing all edges in $\{e_i, e_i'\,\forall i \in I\}$ and the channel gain $m_{e_i;e_i'}(\underline{\mathbf{x}}')$ on G'. Then, $m_{e_i;e_i'}(\underline{\mathbf{x}}')$ can be derived from $m_{e_i;e_i'}(\underline{\mathbf{x}})$ by substituting those $\underline{\mathbf{x}}$ variables that are not in G' by zero. As a result, we immediately have $f(\{m_{e_i;e_i'}(\underline{\mathbf{x}}')\,\forall i \in I\}) = \alpha g(\{m_{e_i;e_i'}(\underline{\mathbf{x}}')\,\forall i \in I\})$ for the same α. The proof of this direction is thus complete. ■

Remark: By Theorem 2, the linear dependence of $\mathbf{h}_i^{(n)}(\underline{\mathbf{x}})$ on $G_{3\text{ANA}}$ is strengthened to for all subgraphs of $G_{3\text{ANA}}$ containing $\{e_{s_i}, e_{d_i}, \forall i\}$, if we carefully select $f(\cdot)$ and $g(\cdot)$. Thus, $\mathbf{h}_i^{(n)}(\underline{\mathbf{x}})$ is linearly independent on $G_{3\text{ANA}}$ *if and only if* there exists a subgraph of $G_{3\text{ANA}}$ containing $\{e_{s_i}, e_{d_i}, \forall i\}$ on which $\mathbf{h}_i^{(n)}(\underline{\mathbf{x}}')$ is linearly independent. This has a similar flavor to the result of the classic framework [5], [7], since for the single multicast from a single source s to the set of destinations $\{d_j\}$, the existence of subgraph induced by the set of edge-disjoint paths from s to d_j is enough to say that the network transfer matrix from s to d_j is full-rank w.h.p.

Theorem 2 can be used to show that (13), (14), and (15) can be satisfied by checking much simpler graph-theoretic conditions.

Corollary 1 (Only $\Rightarrow$ direction proved in [13]): Given a $G_{3\text{ANA}}$, consider the corresponding channel gains $m_{ij}(\underline{\mathbf{x}})$ as defined in Section II-D. Then, $\mathsf{EC}(\{s_{i_1}, s_{i_2}\}; \{d_{j_1}, d_{j_2}\}) = 1$ *if and only if* $m_{i_1j_1}(\underline{\mathbf{x}})m_{i_2j_2}(\underline{\mathbf{x}}) \equiv m_{i_2j_1}(\underline{\mathbf{x}})m_{i_1j_2}(\underline{\mathbf{x}})$.

Proof of $\Rightarrow$*:* It becomes trivial by taking an edge $e \in \mathsf{1cut}(\{s_{i_1}, s_{i_2}\}; \{d_{j_1}, d_{j_2}\})$ and easy manipulation.

Proof of $\Leftarrow$*:* WLOG, $(i_1, i_2) = (1, 2)$ and $(j_1, j_2) = (1, 3)$. Given a $G_{3\text{ANA}}$ with its network variables $\underline{\mathbf{x}}$, suppose by contradiction that $\mathsf{EC}(\{s_1, s_2\}; \{d_1, d_3\}) \geq 2$. Then, by Theorem 2, it suffices to show the existence a subgraph $G' \subseteq G_{3\text{ANA}}$ containing $\{e_{s_1}, e_{s_2}, e_{d_1}, e_{d_3}\}$ which satisfies $m_{11}(\underline{\mathbf{x}}')m_{23}(\underline{\mathbf{x}}') \not\equiv m_{13}(\underline{\mathbf{x}}')m_{21}(\underline{\mathbf{x}}')$. But $\mathsf{EC}(\{s_1, s_2\}; \{d_1, d_3\}) \geq 2$ implies that there must exist two edge-disjoint paths from $\{s_1, s_2\}$ to $\{d_1, d_3\}$ by min-cut/max-flow bound [1]. The subgraph G' induced by such two edge-disjoint paths results in either $m_{11}(\underline{\mathbf{x}}')m_{23}(\underline{\mathbf{x}}') \not\equiv 1$ with $m_{13}(\underline{\mathbf{x}}')m_{21}(\underline{\mathbf{x}}') = 0$ or $m_{11}(\underline{\mathbf{x}}')m_{23}(\underline{\mathbf{x}}') = 0$ with $m_{13}(\underline{\mathbf{x}}')m_{21}(\underline{\mathbf{x}}') \not\equiv 1$ when they are even vertex-disjoint, and results in $m_{11}(\underline{\mathbf{x}}')m_{23}(\underline{\mathbf{x}}') \not\equiv m_{13}(\underline{\mathbf{x}}')m_{21}(\underline{\mathbf{x}}')$ when they share only vertexes. Thus the proof is complete. ■

C. New Channel Gain Property

With Theorems 1 and 2, checking whether $\mathbf{S}_i^{(n)}$ being full-rank w.h.p. on $G_{3\text{ANA}}$ or not can be reduced to finding one subgraph such that the resulting $\mathbf{S}_i^{(n)}$ being full-rank w.h.p. However, the guidance on how to search for such a subgraph of $G_{3\text{ANA}}$ is still missing. To proceed, we need deeper understanding about the channel gain property in the *precoding*-based framework.

Theorem 3 (Channel Gain Property): Given a DAG G and two distinct edges e_s and e_d where $s = \mathsf{head}(e_s)$ and $d = \mathsf{tail}(e_d)$, the following is true (we drop the variables $\underline{\mathbf{x}}$ for shorthand):

- If $\mathsf{EC}(s; d) = 0$, then $m_{e_s;e_d} = 0$
- If $\mathsf{EC}(s; d) = 1$, then $m_{e_s;e_d}$ is reducible and can be expressed as $m_{e_s;e_d} = m_{e_s;e_1}\left(\prod_{i=1}^{N-1} m_{e_i;e_{i+1}}\right)m_{e_N;e_d}$ where $\{e_i\}_{i=1}^N$ are all the distinct 1-*edge cuts* between s and d in the topological order (from the most *upstream* to the most *downstream*). Moreover, the polynomial factors $m_{e_s;e_1}$, $\{m_{e_i;e_{i+1}}\}_{i=1}^{N-1}$, and $m_{e_N;e_d}$ are all irreducible, and no two of them are *equivalent*.
- If $\mathsf{EC}(s; d) \geq 2$ (including ∞), then $m_{e_s;e_d}$ is irreducible.

Proof: Due to space limitations, this proof is omitted. Detailed complete proof can be found in [16]. ■

Remark: Theorem 3 only considers a channel gain between two distinct edges. If $e_s = e_d$, then $m_{e_s;e_d} = 1$ from the convention [5].

Corollary 2: Given a $G_{3\text{ANA}}$, consider the corresponding channel gains $m_{ij}(\underline{\mathbf{x}})$ as defined in Section II-D. Then, $\mathsf{GCD}(m_{i_1j_1}, m_{i_2j_2}) \neq m_{i_2j_2}$ unless $(i_1, j_1) = (i_2, j_2)$, i.e., any channel gain $m_{i_1j_1}$ cannot contain the other $m_{i_2j_2}$.

Proof: Suppose by contradiction that $(i_1, j_1) \neq (i_2, j_2)$ and $\mathsf{GCD}(m_{i_1j_1}, m_{i_2j_2}) = m_{i_2j_2}$. Since they discover the disjoint portion of $G_{3\text{ANA}}$, $m_{i_1j_1} \not\equiv m_{i_2j_2}$. Thus $m_{i_1j_1}$ must be reducible and by Theorem 3, it can be expressed as product of irreducibles, no two of which are *equivalent*. Then, the fact that $m_{i_1j_1}$ contains $m_{i_2j_2}$ as a factor implies that s_{i_2} is in the *downstream* of s_{i_1} if $i_1 \neq i_2$, and d_{j_2} is in the *upstream* of d_{j_1} if $j_1 \neq j_2$, which violates the definition of G_{3ANA}. ■

IV. Detailed Studies of 3-ANA Scheme

In Section III, we investigated the key relationships between the channel gain and the underlying DAG G. For example, Theorem 2 can be used to check whether $L(\underline{\mathbf{x}}) \not\equiv R(\underline{\mathbf{x}})$ holds or not on the G_{3ANA} of interest. That is, if we can find one subgraph containing $\{e_{s_i}, e_{d_i}, \forall\, i\}$ of the given G_{3ANA} on which $L(\underline{\mathbf{x}}) \not\equiv R(\underline{\mathbf{x}})$ holds, then the original G_{3ANA} must also satisfy $L(\underline{\mathbf{x}}) \not\equiv R(\underline{\mathbf{x}})$.

However, Theorem 2 does not provide any graph-theoretic guidance on how to find such a subgraph. In this section, we answer this open challenge by characterizing graph-theoretically the feasibility of the 3-ANA scheme.

A. New Graph-Theoretic Notations and Properties

Consider three indices i, j, and k taking values in $\{1,2,3\}$, for which the values of j and k must be different. Given a G_{3ANA}, let us define:

$$\overline{S}_{i;j\cap k} \triangleq \{\ e \in E\backslash\{e_{s_i}\}\ :\ e \in \mathsf{1cut}(s_i;d_j) \cap \mathsf{1cut}(s_i;d_k)\ \},$$
$$\overline{D}_{i;j\cap k} \triangleq \{\ e \in E\backslash\{e_{d_j}\}\ :\ e \in \mathsf{1cut}(s_j;d_i) \cap \mathsf{1cut}(s_k;d_i)\ \}.$$

When the values of indices i, j, and k are all different, we use $\overline{S}_i$ (resp. $\overline{D}_i$) as shorthand for $\overline{S}_{i;j\cap k}$ (resp. $\overline{D}_{i;j\cap k}$). Then, the following lemmas show some topological relationships among them. We will assume that the values of indices i, j, and k used in the following lemmas are all different unless noted. Due to space limitations, the proofs to the following lemmas are omitted. Please see [16].

Lemma 1: If $e' \in \overline{S}_i$ and $e'' \in \overline{D}_j$, then one of the following statements is true: $e' \prec e''$, $e' \succ e''$, or $e' = e''$.

Lemma 2: $(\overline{S}_i \cap \overline{S}_j) \subseteq \overline{D}_k$.

Lemma 3: For all $e' \in \overline{S}_i \backslash \overline{D}_j$ and all $e'' \in \overline{D}_j$, $e' \preceq e''$.

Lemma 4: $\overline{D}_j \cap \overline{D}_k \neq \emptyset$ *if and only if* both $\overline{S}_i \cap \overline{D}_j \neq \emptyset$ and $\overline{S}_i \cap \overline{D}_k \neq \emptyset$.

Lemma 5: For all $e'' \in \overline{D}_i \cap \overline{D}_j$, if $\overline{S}_i \cap \overline{S}_j \neq \emptyset$, then there exists $e'_* \in \overline{S}_i \cap \overline{S}_j$ such that $e'_* \preceq e''$.

Lemma 6: Consider four indices i, j_1, j_2, and j_3 taking values in $\{1,2,3\}$ for which the values of j_1, j_2 and j_3 must be all different. If $\overline{S}_{i;j_1\cap j_2} \neq \emptyset$ and $\overline{S}_{i;j_1\cap j_3} \neq \emptyset$, then $\overline{S}_{i;j_1\cap j_2} \cap \overline{S}_{i;j_1\cap j_3} \neq \emptyset$ thereby $\overline{S}_{i;j_2\cap j_3} \neq \emptyset$ and $\overline{S}_i \neq \emptyset$.

Remark: If we swap the roles of $\overline{S}$ and $\overline{D}$, then the corresponding statements from Lemma 1 to Lemma 6 still hold. For example, Lemma 2 also implies $(\overline{D}_i \cap \overline{D}_j) \subseteq \overline{S}_k$.

We also prove some relationship between the channel gains on G_{3ANA} and the 1-*edge cuts*.

Lemma 7: Given a G_{3ANA}, consider the corresponding channel gains as defined in Section II-D. Consider three indices i, j_1, and j_2 taking values in $\{1,2,3\}$ for which the values of j_1 and j_2 must be different. If $\mathsf{GCD}(\,m_{ij_1}, m_{ij_2}) \not\equiv 1$, then $\overline{S}_{i;j_1\cap j_2} \neq \emptyset$. (Similarly, if $\mathsf{GCD}(\,m_{j_1i}, m_{j_2i}) \not\equiv 1$, then $\overline{D}_{i;j_1\cap j_2} \neq \emptyset$.)

B. Characterizing the GTC of $L(\underline{\mathbf{x}}) \not\equiv R(\underline{\mathbf{x}})$

We first prove the following graph-theoretic condition which implies $L(\underline{\mathbf{x}}) \not\equiv R(\underline{\mathbf{x}})$.

Proposition 1: If there exists a pair of indices $i, j \in \{1,2,3\}$ where $i \neq j$ satisfying both $\overline{S}_i \cap \overline{S}_j \neq \emptyset$ and $\overline{D}_i \cap \overline{D}_j \neq \emptyset$ on a given G_{3ANA}, then we have $L(\underline{\mathbf{x}}) \equiv R(\underline{\mathbf{x}})$.

Proof: WLOG, suppose $\overline{S}_1 \cap \overline{S}_2 \neq \emptyset$ and $\overline{D}_1 \cap \overline{D}_2 \neq \emptyset$ ($i = 1$ and $j = 2$). By Lemma 5, we can find two edges $e_1 \in \overline{S}_1 \cap \overline{S}_2$ and $e_2 \in \overline{D}_1 \cap \overline{D}_2$ such that $e_1 \preceq e_2$. Also note that $e_1 \in \overline{D}_3$ and $e_2 \in \overline{S}_3$ by Lemma 2. Then by Theorem 3, the channel gains $m_{ij}(\underline{\mathbf{x}}), i \neq j$ can be expressed by (we drop $\underline{\mathbf{x}}$ for shorthand):

$$m_{13} = m_{e_{s_1};e_1}\, m_{e_1;e_{d_3}} \qquad m_{12} = m_{e_{s_1};e_1}\, m_{e_1;e_2}\, m_{e_2;e_{d_2}}$$
$$m_{32} = m_{e_{s_3};e_2}\, m_{e_2;e_{d_2}} \qquad m_{23} = m_{e_{s_2};e_1}\, m_{e_1;e_{d_3}}$$
$$m_{21} = m_{e_{s_2};e_1}\, m_{e_1;e_2}\, m_{e_2;e_{d_1}} \qquad m_{31} = m_{e_{s_3};e_2}\, m_{e_2;e_{d_1}}$$

where the expressions of m_{12} and m_{21} are derived based on Theorem 3, and the facts that $e_1 \preceq e_2$ and both e_1 and e_2 belong to $\mathsf{1cut}(s_1;d_2) \cap \mathsf{1cut}(s_2;d_1)$ for example. We can easily verify that the G_{3ANA} of interest satisfies $L \equiv R$. ■

Remark: In the example of Fig. 1, one can easily see that $e_1 \in \overline{S}_1 \cap \overline{S}_2$ and $e_2 \in \overline{D}_1 \cap \overline{D}_2$. Hence, Proposition 1 proves that the example network of Fig. 1 satisfies $L(\underline{\mathbf{x}}) \equiv R(\underline{\mathbf{x}})$ without actually computing $L(\underline{\mathbf{x}})$ and $R(\underline{\mathbf{x}})$.

For the following, we will show that the graph-theoretic condition identified in Proposition 1 is also necessary for $L(\underline{\mathbf{x}}) \equiv R(\underline{\mathbf{x}})$. Before proceeding, we prove the following graph-theoretic properties about the channel gains conditioning on $L(\underline{\mathbf{x}}) \equiv R(\underline{\mathbf{x}})$. We drop $\underline{\mathbf{x}}$ for shorthand.

Lemma 8: If the G_{3ANA} of interest satisfies $L \equiv R$, then $\overline{S}_i \neq \emptyset$ and $\overline{D}_j \neq \emptyset$ for all i and j, respectively.

Proof: Suppose the G_{3ANA} of interest satisfies $L \equiv R$. We first prove that $\mathsf{EC}(\mathsf{head}(e_{s_i});\mathsf{tail}(e_{d_j})) = 1$ for all $i \neq j$. Suppose by contradiction there exists a pair $i \neq j$ such that $\mathsf{EC}(\mathsf{head}(e_{s_i});\mathsf{tail}(e_{d_j})) \neq 1$. WLOG, assume $\mathsf{EC}(\mathsf{head}(e_{s_1});\mathsf{tail}(e_{d_2})) \neq 1$ (i.e., $i = 1$ and $j = 2$). Since we are focusing on a 3-ANA network, we must have $\mathsf{EC}(\mathsf{head}(e_{s_1});\mathsf{tail}(e_{d_2})) \geq 2$. By Theorem 3, m_{12} is irreducible. However, since $L \equiv R$, m_{12} must be a factor of (at least) one of the three polynomials m_{13}, m_{32}, or m_{21}. This, however, contradicts Corollary 2. As a result, $\mathsf{EC}(\mathsf{head}(e_{s_i});\mathsf{tail}(e_{d_j})) = 1$ for all $i \neq j$. This shows that for all $i \neq j$, we can decompose the channel gain m_{ij} by

$$m_{ij} = m_{e_{s_i};e_1^{ij}} \left(\prod_{k=1}^{N_{ij}-1} m_{e_k^{ij};e_{k+1}^{ij}} \right) m_{e_{N_{ij}}^{ij};e_{d_j}} \qquad (16)$$

where N_{ij} is the number of 1-*edge cuts* separating s_i and d_j; $\{e_k^{ij}\}_{k=1}^{N_{ij}}$ list all those 1-*edge cuts* from the most *upstream* to the most *downstream* one but not counting e_{s_i} and e_{d_j}; and $m_{e_{s_i};e_1^{ij}}$, $\{m_{e_k^{ij};e_{k+1}^{ij}}\}_{k=1}^{N_{ij}-1}$, and $m_{e_{N_{ij}}^{ij};e_{d_j}}$ are all irreducible polynomial and no two of them are *equivalent* to each other.

We now show that for any three distinct index values i_1, i_2, and j, we must have $\mathsf{GCD}(\,m_{i_1j}, m_{i_2j}) \not\equiv 1$. We prove this statement by contradiction. Suppose, say $\mathsf{GCD}(\,m_{21}, m_{31}) \not\equiv 1$. Then by the assumption that $L \equiv R$, we must have

$\mathsf{GCD}(m_{21}, m_{12}m_{23}) = m_{21}$. As a result, all the irreducible factors of m_{21}, see (16), must also be factors of $m_{12}m_{23}$. For example, the irreducible factor $m_{e^{21}_{N_{21}};e_{d_1}}$ of m_{21} must be a factor of $m_{12}m_{23}$. By Theorem 3, this is possible only when e_{d_1} is a 1-*edge cut* separating either $\{s_1\}$ and $\{d_2\}$ or $\{s_2\}$ and $\{d_3\}$. However, this contradicts the 3-ANA network assumption that $|\mathsf{Out}(d_1)| = 0$.

By Lemma 7 and the above discussion, we must have $\overline{D}_j \neq \emptyset$ for all j. By repeating the symmetric arguments for $\overline{S}_i$, the proof is complete. ■

We now prove the necessity counterpart of Proposition 1.

Proposition 2: If the G_{3ANA} of interest satisfies $L(\underline{\mathbf{x}}) \equiv R(\underline{\mathbf{x}})$, then there exists a pair of indices $i, j \in \{1, 2, 3\}$ where $i \neq j$ satisfying both $\overline{S}_i \cap \overline{S}_j \neq \emptyset$ and $\overline{D}_i \cap \overline{D}_j \neq \emptyset$.

Proof: Suppose the G_{3ANA} of interest satisfies $L \equiv R$. By Lemma 8, we knew that each channel gain m_{ij} $(i \neq j)$ in the expression of $L \equiv R$ can be expressed as in (16). In addition, $\overline{S}_i \neq \emptyset$ and $\overline{D}_j \neq \emptyset$ for all i and j.

Case 1 when $\overline{S}_i \cap \overline{D}_j = \emptyset$ for some $i \neq j$:

WLOG, assume $\overline{S}_2 \cap \overline{D}_1 = \emptyset$ ($i = 2$ and $j = 1$). Let e_2^* denote the most *downstream* edge in $\overline{S}_2$ and let e_1^* denote the most *upstream* edge in $\overline{D}_1$. Since $\overline{S}_2 \cap \overline{D}_1 = \emptyset$, edge e_2^* must not be in $\overline{D}_1$. By Lemma 3, we must have $e_2^* \prec e_1^*$.

We fisrt show that $\{e_2^*, e_1^*\} \subset$ 1cut$(s_1; d_2)$. We first notice that by definition, $e_2^* \in \overline{S}_2 \subseteq \mathsf{1cut}(s_2; d_1)$ and $e_1^* \in \overline{D}_1 \subseteq \mathsf{1cut}(s_2; d_1)$. Therefore, when rewriting m_{21} by (16), both e_2^* and e_1^* must participate in the form of $e_2^* = e^{21}_{\hat{N}_{21}}$ and $e_1^* = e^{21}_{\tilde{N}_{21}}$ for two integers $\hat{N}_{21}$ and $\tilde{N}_{21}$ satisfying $1 \leq \hat{N}_{21} < \tilde{N}_{21} \leq N_{21}$. Define temporarily:

$$m_{e_2^*;e_1^*} = \prod_{k=\hat{N}_{21}}^{\tilde{N}_{21}-1} m_{e^{21}_k;e^{21}_{k+1}}$$

where $m_{e_2^*;e_1^*} \not\equiv 1$ by our construction of $e_2^* \prec e_1^*$.

We now claim that $\mathsf{GCD}(m_{e_2^*;e_1^*}, m_{23}m_{31}) \equiv 1$, i.e. $m_{23}m_{31}$ cannot contain any irreducible factor of $m_{e_2^*;e_1^*}$. Suppose by contradiction that m_{23} contains any irreducible factor of $m_{e_2^*;e_1^*}$, say $m_{e^{21}_{\bar{k}};e^{21}_{\bar{k}+1}}$ where $\hat{N}_{21} \leq \bar{k} \leq \tilde{N}_{21} - 1$. Then, by Theorem 3, $e^{21}_{\bar{k}+1}$ must belong to $\mathsf{1cut}(s_2; d_3)$. Since $e^{21}_{\bar{k}+1} \in \mathsf{1cut}(s_2; d_1)$, this implies that $e^{21}_{\bar{k}+1} \in \overline{S}_2$. This, however, contradicts the assumption that $e_2^* = e^{21}_{\hat{N}_{21}} \prec e^{21}_{\bar{k}+1}$ is the most *downstream* edge in $\overline{S}_2$. As a result, m_{23} must not contain any irreducible factor of $m_{e_2^*;e_1^*}$. By a symmetric argument, we can also show that m_{31} must not contain any irreducible factor of $m_{e_2^*;e_1^*}$. Since the assumption of $L \equiv R$ implies that $\mathsf{GCD}(m_{e_2^*;e_1^*}, R) = m_{e_2^*;e_1^*}$, we thus must have $\mathsf{GCD}(m_{e_2^*;e_1^*}, m_{12}) = m_{e_2^*;e_1^*}$. Note that $m_{12} \not\equiv m_{e_2^*;e_1^*}$, otherwise s_1 will be a *downstream* node of s_2. By Theorem 3, both e_1^* and e_2^* must belong to $\mathsf{1cut}(s_1; d_2)$.

We prove that $e_2^* \in$ 1cut$(s_1; d_3)$ and $e_1^* \in$ 1cut$(s_3; d_2)$. Take an arbitrary P_{21} path from s_2 to d_1. Since $\{e_2^*, e_1^*\} \subset \mathsf{1cut}(s_2; d_1)$, both e_2^* and e_1^* must be used by P_{21}. Since $e_2^* \in \mathsf{1cut}(s_1; d_2)$ implies $s_1 \prec \mathsf{tail}(e_2^*)$, and $e_2^* \in \overline{S}_2$ implies $\mathsf{head}(e_2^*) \prec d_3$, we can construct an arbitrary path P_{13} from s_1 to d_3 passing through e_2^*. Furthermore, since $e_1^* \in \overline{D}_1$ implies $s_3 \prec \mathsf{tail}(e_1^*)$, and $e_1^* \in \mathsf{1cut}(s_1; d_2)$ implies $\mathsf{head}(e_1^*) \prec d_2$, we can construct an arbitrary path P_{32} from s_3 to d_2 passing through e_1^*. Then, using these constructed paths P_{21}, P_{13}, and P_{32} and the fact that both e_2^* and e_1^* belong to $\mathsf{1cut}(s_1; d_2)$, we will prove that $e_2^* \in \mathsf{1cut}(s_1; d_3)$ and $e_1^* \in \mathsf{1cut}(s_3; d_2)$.

We show that $e_2^* \in \mathsf{1cut}(s_1; d_3)$. By contradiction, suppose that there exists a path Q_{13} from s_1 to d_3 not passing through e_2^* (assume $e_2^* \notin \mathsf{1cut}(s_1; d_3)$). Consider P_{32} and P_{21} above. As constructed before, e_1^* is used by both paths. Consider the set of nodes that both paths pass through (including $\mathsf{tail}(e_1^*)$ and $\mathsf{head}(e_1^*)$), and denote its most *downstream* node as w. In addition, denote $e_w \in P_{21}$ where $\mathsf{tail}(e_w) = w$. WLOG, let $w \succ \mathsf{head}(e_1^*)$. Then, Q_{13} must be vertex-disjoint with both the path segment $\mathsf{head}(e_2^*)P_{21}w$ and P_{32}, otherwise we can construct a path from s_1 to d_2 not passing through e_2^*, which violates $e_2^* \in \mathsf{1cut}(s_1; d_2)$. Furthermore, Q_{13} must be vertex-disjoint with the path segment $s_2P_{21}\mathsf{tail}(e_2^*)$, otherwise we can construct a path from s_2 to d_3 not passing through e_2^*, which violates $e_2^* \in \overline{S}_2$. However, Q_{13} may or may not be vertex-disjoint with the path segment $\mathsf{head}(e_w)P_{21}d_1$. Then, the subgraph induced by $\{Q_{13}, P_{32}, P_{21}\}$ (where $L \not\equiv 1$) satisfies $L \not\equiv R$ because there are no paths from s_1 to d_2 resulting $m_{12}(\underline{\mathbf{x}}') = 0$ thereby $R = 0$ on that subgraph, which contradicts the assumption that G_{3ANA} satisfies $L \equiv R$ by Theorem 2. Thus, $e_2^* \in \mathsf{1cut}(s_1; d_3)$.

By a similar argument using the constructed paths P_{13} and P_{21}, and any arbitrary path Q_{32} from s_3 to d_2 not passing through e_1^*, we can prove that $e_1^* \in \mathsf{1cut}(s_3; d_2)$.

Therefore, with the argument that $e_2^* \in \mathsf{1cut}(s_1; d_2)$ and $e_1^* \in \mathsf{1cut}(s_1; d_2)$ when G_{3ANA} satisfies $L \equiv R$ and $\overline{S}_2 \cap \overline{D}_1 = \emptyset$, we showed that $e_2^* \in \mathsf{1cut}(s_1; d_3)$ and $e_1^* \in \mathsf{1cut}(s_3; d_2)$, which implies that $e_2^* \in \overline{S}_1$ and $e_1^* \in \overline{D}_2$. Since $e_2^* \in \overline{S}_2$ and $e_1^* \in \overline{D}_1$ by assumption, we prove that $\overline{S}_1 \cap \overline{S}_2 \neq \emptyset$ and $\overline{D}_1 \cap \overline{D}_2 \neq \emptyset$. The proof of *Case 1* is thus complete.

Case 2 when $\overline{S}_i \cap \overline{D}_j \neq \emptyset$ for all $i \neq j$:

By Lemma 4, we must have $\overline{S}_i \cap \overline{S}_j \neq \emptyset$ and $\overline{D}_i \cap \overline{D}_j \neq \emptyset$ $\forall i \neq j$. The proof of *Case 2* is complete. ■

C. GTC of the Feasibility of 3-ANA with $n = 1$

Propositions 1 and 2 provide the graph-theoretic condition that characterizes whether the G_{3ANA} of interest satisfies $L(\underline{\mathbf{x}}) \not\equiv R(\underline{\mathbf{x}})$ or not. However, to ensure the feasibility of the 3-ANA scheme, $\mathbf{h}_i^{(n)}(\underline{\mathbf{x}})$ must be linearly independent on G_{3ANA} for all i. In this subsection, we prove a graph-theoretic characterization characterizing the linear independence of $\mathbf{h}_i^{(n)}(\underline{\mathbf{x}})$ for the simplest case of $n = 1$.

Consider the following graph-theoretic conditions:

$$\overline{S}_i \cap \overline{S}_j = \emptyset \text{ or } \overline{D}_i \cap \overline{D}_j = \emptyset \;\; \forall \{i,j\} \subset \{1,2,3\} \quad (17)$$

$$\mathsf{EC}(\{s_1,s_2\};\{d_1,d_3\}) \geq 2,\; \mathsf{EC}(\{s_1,s_3\};\{d_1,d_2\}) \geq 2 \quad (18)$$

$$\mathsf{EC}(\{s_1,s_2\};\{d_2,d_3\}) \geq 2,\; \mathsf{EC}(\{s_2,s_3\};\{d_1,d_2\}) \geq 2 \quad (19)$$

$$\mathsf{EC}(\{s_1,s_3\};\{d_2,d_3\}) \geq 2,\; \mathsf{EC}(\{s_2,s_3\};\{d_1,d_3\}) \geq 2 \quad (20)$$

Proposition 3: 3-ANA with $n = 1$ is feasible on the G_{3ANA} of interest *if and only if* G_{3ANA} satisfies (17-20).

Proof: By Proposition 1 and 2, the first feasibility condition of 3-unicast ANA (G_{3ANA} satisfies $L \not\equiv R$) is equivalent to (17). By Theorem 1, $\det(\mathbf{S}_i^{(n)}) \neq 0 \,\forall\, i$ w.h.p. whenever $\mathbf{h}_1^{(n)}(\underline{\mathbf{x}})$, $\mathbf{h}_2^{(n)}(\underline{\mathbf{x}})$, and $\mathbf{h}_3^{(n)}(\underline{\mathbf{x}})$ is linearly independent on G_{3ANA}, respectively. Also by Corollary 1, G_{3ANA} satisfies both $m_{11}m_{23} \not\equiv m_{13}m_{21}$ and $m_{11}m_{32} \not\equiv m_{12}m_{31}$ *if and only if* (18) holds on G_{3ANA}. (Similarly, jointly $m_{22}m_{13} \not\equiv m_{23}m_{12}$ and $m_{22}m_{31} \not\equiv m_{21}m_{32}$ are equivalent to (19), and jointly $m_{33}m_{12} \not\equiv m_{32}m_{13}$ and $m_{33}m_{21} \not\equiv m_{31}m_{23}$ are equivalent to (20).)

Thus for a G_{3ANA} satisfying $L \not\equiv R$, we need to show that,

(a) $\mathbf{h}_1^{(n)}(\underline{\mathbf{x}})$ is linearly independent *if and only if* $m_{11}m_{23} \not\equiv m_{13}m_{21}$ and $m_{11}m_{32} \not\equiv m_{12}m_{31}$.
(b) $\mathbf{h}_2^{(n)}(\underline{\mathbf{x}})$ is linearly independent *if and only if* $m_{22}m_{13} \not\equiv m_{23}m_{12}$ and $m_{22}m_{31} \not\equiv m_{21}m_{32}$.
(c) $\mathbf{h}_3^{(n)}(\underline{\mathbf{x}})$ is linearly independent *if and only if* $m_{33}m_{12} \not\equiv m_{32}m_{13}$ and $m_{33}m_{21} \not\equiv m_{31}m_{23}$.

We prove only (a). The proof for (b) and (c) will be followed similarly. When $n = 1$,

$$
\begin{aligned}
\mathbf{h}_1^{(1)}(\underline{\mathbf{x}}) &= \{m_{11}\mathbf{v}_1^{(1)},\ m_{21}\mathbf{v}_2^{(1)}\} \\
&= \{m_{11}m_{23}m_{32}R,\ m_{11}m_{23}m_{32}L,\ m_{21}m_{13}m_{32}R\}
\end{aligned}
$$

Proof of (a), $\Rightarrow$*:* By contradiction, suppose G_{3ANA} satisfies either $m_{11}m_{23} \equiv m_{13}m_{21}$ or $m_{11}m_{32} \equiv m_{12}m_{31}$ or both. Then, it is easy to see that $\mathbf{h}_1^{(1)}(\underline{\mathbf{x}})$ is linearly dependent.

Proof of (a), $\Leftarrow$*:* This proof is omitted due to space limitations. Please refer to [16]. ■

The problem of finding the graph-theoretic condition for general $n \geq 2$ remains an open problem. On the other hand, we prove the following corollary which shows that the feasibility conditions for $n = 1$ case turns out to be necessary for the cases of general n values.

Corollary 3: For the G_{3ANA} of interest, if 3-ANA with $n \geq 2$ is feasible on G_{3ANA}, then 3-ANA with $n = 1$ is also feasible on G_{3ANA}.

Proof: If 3-ANA with $n \geq 2$ is feasible on a G_{3ANA}, then $\mathbf{h}_i^{(n)}(\underline{\mathbf{x}})$ is linearly independent on G_{3ANA} for $i \in \{1, 2, 3\}$ by Theorem 1, along with that G_{3ANA} satisfies $L(\underline{\mathbf{x}}) \not\equiv R(\underline{\mathbf{x}})$. If $\mathbf{h}_i^{(n)}(\underline{\mathbf{x}})$ is linearly independent, then $\mathbf{h}_i^{(1)}(\underline{\mathbf{x}})$, a subset of the original polynomials, is linearly independent as well. By Theorem 1, 3-ANA with $n = 1$ is feasible on that G_{3ANA}. ■

Fig. 2 summarizes our knowledge about the graph-theoretic characterization of the 3-ANA scheme, in which three arrows have been established, except the right one. That is, we have developed necessary and sufficient graph-theoretic condition for the 3-ANA schemes with $n = 1$. In our future work, we will investigate graph-theoretic characterization problem for general $n \geq 2$.

V. Conclusion and Future Works

The main subject of this work is the general class of *precoding*-based NC schemes, which focus on designing the precoding and decoding mappings at the sources and destinations while using randomly generated *local encoding kernels* within the network. Such a precoding-based structure includes the 3-ANA scheme, originally proposed in [12], [13], as a special case. In this work, we have identified new

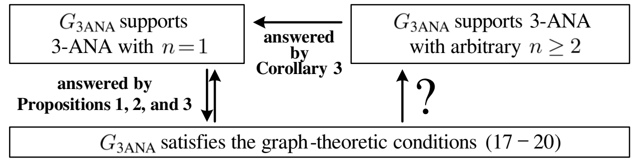

Fig. 2. Current understanding of 3-ANA graph-theoretic characterization.

graph-theoretic vs algebraic relationship for the *precoding*-based NC solutions. Based on the findings on the general *precoding*-based NC, we have further characterized the graph-theoretic feasibility conditions of the 3-ANA scheme for the simplest case of $n = 1$, which includes proving the open conjectures of the existing results for $n = 1$. We believe that the fundamental analysis in this work will serve as a precursor to fully understand the notoriously challenging multiple-unicast NC problem and design practical, distributed NC solutions based on the *precoding*-based framework.

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